THE CHARACTERISTIC POLYNOMIAL OF MULTI-ROOTED, DIRECTED TREES

LAUREN EATON

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ABSTRACT. We define the characteristic polynomial for single-rooted trees and begin with a theorem about this polynomial, derived from the known contraction/deletion formula. We expand our scope to include multi-rooted, directed trees. Introducing the concept of a star, we prove two theorems which allow us to evaluate the polynomials of these trees in terms of the stars that comprise them. Finally, we derive and prove a general formula for the characteristic polynomial of multi-rooted, directed trees.

INTRODUCTION

Introduced by Gary Gordon and Elizabeth McMahon in [3] the greedoid characteristic polynomial is a generalization of the Tutte polynomial, which is a two-variable invariant which helps describe the structure of a single-rooted graph. Given a multi-rooted, directed tree, one must conduct an often tedious recursive calculation in order to obtain the polynomial. In this paper, we work towards defining a general formula for the characteristic polynomial that requires minimal calculation and is entirely in terms of the vertices.

This paper is organized as follows: In Section 1, we start with a focus on single-rooted trees and an adaptation of the characteristic polynomial.

In Section 2, we look at the more complex multi-rooted, directed trees. We will find that when deriving the characteristic polynomial of these complicated trees, one can consider them in simplified terms. We will explore the concept of a *sink* and see how the polynomial is affected by its presence. Likewise, we introduce a graphical element called a *star*; a multi-rooted, directed tree can be thought of as being composed entirely of stars with shared edges. We show the polynomial of this tree is equivalent to a product whose terms are polynomials of the stars that it is composed of. This leads to our final theorem, a formula for calculating the polynomial of a multi-rooted, directed tree in terms of the contributions of its vertices.

1. Single-Rooted Trees

Definitions. In this paper, we expand upon published research conducted by Gary Gordon and Elizabeth McMahon [1], [2], [3] and thus reference their work throughout. We recall various definitions from graph theory, first those pertinent and relating to single-rooted trees and then those related to multi-rooted trees.

Let $G = (V, E, r)$ be a rooted *graph*. A *graph* G is a finite nonempty set of vertices together with a set of unordered pairs of distinct vertices of G called *edges*. Specifically, G is a *single-rooted tree* with edge set $E(G)$, vertex set $V(G)$, and root r, where a *root* is any distinguished vertex. A *tree* is defined as a graph in which any two vertices are connected by exactly one *path*. A *path* is a finite, alternating sequence of incident vertices and edges in which no vertex is repeated. A tree is a *connected* graph, because, given any $u, v \in V$, there exists a path between u and v in G. A *tree* is also an *acyclic connected* graph. An *acyclic graph* contains no *cycle* — a nontrivial closed finite, alternating sequence of vertices and edges in which no edge is repeated. The *degree* of a vertex v in G, denoted $\deg_G(v)$, is the number of edges incident with v. Define L to be the set of *leaves* — vertices, excluding roots, with degree equal to 1— $L = \{v \in V - r : \deg_G(v) = 1\}$, and let $\ell = |L|$. For further definitions, see [5].

Let G_i be a *subgraph* of G, denoted $G_i \subseteq G$. A graph G_i is a *subgraph* of G if $V(G_i) \subseteq V(G)$ and $E(G_i) \subseteq E(G)$. Also, the root r is an element of G_i.

Definition 1. If G_1 and G_2 are subgraphs of G such that $E(G_1) \bigcap E(G_2) = \emptyset$, $V(G_1) \bigcap V(G_2) = \{r\}$, $E(G_1) \bigcup E(G_2) = E(G)$ and $V(G_1) \bigcup V(G_2) = V(G)$, then

G is the *direct sum* of G_1 and G_2, and we denote $G = G_1 \oplus G_2$.

We use E and $E(G)$ interchangeably. Similarly, $E_i = E(G_i)$. Suppose $S \subseteq E$.

Definition 2. Let $r_G(S)$ denotes the *rank* of S in G: the number of vertices connected to the root by some path in S. The rank does not include the root. Similarly, for $G_i \subseteq G$ and $S_i = S \cap E_i$, let $r_{G_i}(S_i)$ correspond to the rank of S_i in G_i.

Suppose $G = G_1 \oplus G_2$; then, E_1 and E_2 form a partition on E with $r_G(E) = r_{G_1}(E_1)+r_{G_2}(E_2)$. Define $S_1 = S\cap E_1$ and $S_2 = S\cap E_2$, so $S = S_1\cup S_2$ and $S_1 \subseteq E_1$ and $S_2 \subseteq E_2$. It follows that $r_G(S) = r_{G_1}(S_1) + r_{G_2}(S_2)$. Let $|S|$ be defined as the number of elements in the set S. For conciseness we define the *corank* of S to be $\mathrm{cor}_G(S) = r_G(E) - r_G(S)$ and $\mathrm{cor}_{G_i}(S_i) = r_{G_i}(E_i) - r_{G_i}(S_i)$. Likewise, we define the *nullity* of S to be $\mathrm{nul}_G(S) = |S| - r_G(S)$ and $\mathrm{nul}_{G_i}(S_i) = |S| - r_{G_i}(S_i)$. As defined by Gary Gordon and Elizabeth McMahon, the *characteristic polynomial* $p(G; \lambda)$ is expressed, in terms of the corank and nullity, by:

Proposition 1 (characteristic polynomial; [1])**.**

$$p(G; \lambda) = (-1)^{r_G(E)} \sum_{S\subseteq E} (-\lambda)^{\mathrm{cor}_G(S)}(-1)^{\mathrm{nul}_G(S)}$$

In the first theorem, we will demonstrate an alternate way of expressing the characteristic polynomial as identified above. Also defined by Gordon and McMahon is the Deletion-Contraction Formula for the characteristic polynomial and the Direct Sum Property, which we will make use of later.

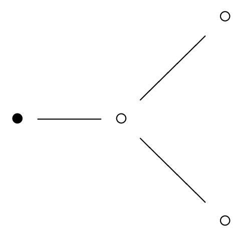
Let e be an edge incident to the root. The *deletion* of e means the erasing it from the graph to obtain new edge set $E - e$ and vertex set $V - r$ where r is the root from which e emanates. The *contraction* of e means erasing e and merging the two endpoints of e to obtain new edge set $E - e$. Since e emanates from a root, we merge the endpoints to form a new root.

Proposition 2 (Deletion-contraction; Prop. 3 [3])**.**

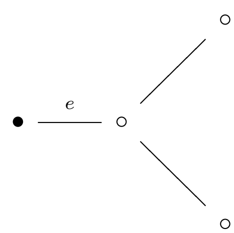
$$p(G;\lambda) = \lambda^{r_G(E)-r_G(E-e)}p(G-e;\lambda) - p(G/e;\lambda).$$

The following is an example of the computation of a polynomial of a single-rooted tree using the deletion-contraction formula.

$\Gamma =$

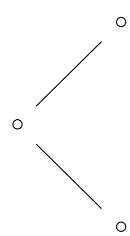


$p(\Gamma;\lambda) = p$ (

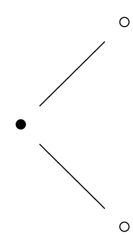


)

$= \lambda^3 \cdot p$ (



) $- p$ (



)

$$= \underbrace{0}_{\text{We will justify this step later}} - \underbrace{(-1)^2(1-\lambda)^2}_{\text{By Direct Sum Property}}$$

$$= (-1)^3(1-\lambda)^2.$$

Also, if $G = G_1 \oplus G_2$, then:

Proposition 3 (Direct Sum Property; Prop. 4 [3])**.**

$$p(G_1 \oplus G_2) = p(G_1) \cdot p(G_2).$$

We begin with the proof of our first theorem, a restating of the characteristic polynomial in Proposition 1.

Theorem 1. *The characteristic polynomial of a single-rooted tree G is given by*

$$p(G;\lambda) = (-1)^{|E|}(1-\lambda)^{\ell}.$$

In order to prove Theorem 1, we begin by proving Proposition 3.

Lemma 1. *For G_i as defined above with $G = G_1 \oplus G_2$,*

$$p(G;\lambda) = p(G_1;\lambda) \cdot p(G_2;\lambda).$$

Proof. Let $X(S) = (-\lambda)^{\mathrm{cor}_G(S)}(-1)^{\mathrm{nul}_G(S)}$ and $X_i(S_i) = (-\lambda)^{\mathrm{cor}_{G_i}(S_i)}(-1)^{\mathrm{nul}_{G_i}(S_i)}$ for $i = 1, 2$. In order to show $X(S) = X_1(S_1) \cdot X_2(S_2)$, we will show $\mathrm{cor}_G(S) = \mathrm{cor}_{G_1}(S_1) + \mathrm{cor}_{G_2}(S_2)$ and $\mathrm{nul}_G(S) = \mathrm{nul}_{G_1}(S_1) + \mathrm{nul}_{G_2}(S_2)$.

First, consider

$$\begin{aligned}
\mathrm{cor}_{G_1}(S_1) + \mathrm{cor}_{G_2}(S_2) &= r_{G_1}(E_1) - r_{G_1}(S_1) + r_{G_2}(E_2) - r_{G_2}(S_2) \\
&= r_{G_1}(E_1) + r_{G_2}(E_2) - (r_{G_1}(S_1) + r_{G_2}(S_2)) \\
&= r_G(E) - r_G(S) \\
&= \mathrm{cor}_G(S).
\end{aligned}$$

Now, consider

$$\begin{aligned}
\mathrm{nul}_{G_1}(S_1) + \mathrm{nul}_{G_2}(S_2) &= |S_1| - r_{G_1}(S_1) + |S_2| - r_{G_2}(S_2) \\
&= |S_1| + |S_2| - (r_{G_1}(S_1) + r_{G_2}(S_2)) \\
&= |S| - r_G(S) \\
&= \mathrm{nul}_G(S).
\end{aligned}$$

Then, $X_1(S_1) \cdot X_2(S_2)$ simplifies as follows:

$$\begin{aligned} X_1(S_1) \cdot X_2(S_2) &= (-\lambda)^{\mathrm{cor}_{G_1}(S_1)}(-\lambda)^{\mathrm{cor}_{G_2}(S_2)}(-1)^{\mathrm{nul}_{G_1}(S_1)}(-1)^{\mathrm{nul}_{G_2}(S_2)} \\ &= (-\lambda)^{\mathrm{cor}_{G_1}(S_1)+\mathrm{cor}_{G_2}(S_2)}(-1)^{\mathrm{nul}_{G_1}(S_1)+\mathrm{nul}_{G_2}(S_2)} \\ &= (-\lambda)^{\mathrm{cor}_{G}(S)}(-1)^{\mathrm{nul}_{G}(S)} \\ &= X(S). \end{aligned}$$

Finally, taking into account $X_1(S_1) \cdot X_2(S_2) = X(S)$, examine $p(G;\lambda) = \sum_{S \subseteq E} X(S)$ as follows:

$$\begin{aligned} p(G;\lambda) &= \sum_{S \subseteq E} X(S) \\ &= \sum_{S_1 \cup S_2 \subseteq E} X(S_1 \cup S_2) \\ &= \sum_{S_1 \subseteq E_1} \left(\sum_{S_2 \subseteq E_2} X(S_1 \cup S_2) \right) \\ &= \sum_{S_1 \subseteq E_1} \left(\sum_{S_2 \subseteq E_2} X_1(S_1) \cdot X_2(S_2) \right) \\ &= \left(\sum_{S_1 \subseteq E_1} X_1(S_1) \right) \left(\sum_{S_2 \subseteq E_2} X_2(S_2) \right) \\ &= p(G_1;\lambda) \cdot p(G_2;\lambda). \end{aligned}$$

Therefore, $p(G;\lambda) = p(G_1;\lambda) \cdot p(G_2;\lambda)$ for all single-rooted trees $G = G_1 \oplus G_2$. □

Now, we will use Lemma 1 in order to prove Theorem 1. We define a *g-loop* to be an edge that is either a *loop* or not connected to a root. A *loop* is an edge that joins a vertex to itself.

Proof of Theorem 1 by Induction. Let $G = (V, E, r)$ be a rooted tree. Let $|E| = k$ for $k \in \mathbb{N}$. We will induct on k.

Base Case: Suppose $k = 1$ and thus $|E| = 1$. It is easily shown that the polynomial for a single-rooted tree G with $|E| = 1$ is $p(G;\lambda) = (-1)^1(1-\lambda)^1$. Using the corank/nullity definition, the following illustrates the base case:

$p(G;\lambda) = p\Big($ $\Big)$

$$\begin{aligned} &= (-1)^{r_G(E)} \sum_{S\subseteq E} (-\lambda)^{\mathrm{cor}_G(S)}(-1)^{\mathrm{nul}_G(S)} \\ &= (-1)^1 \cdot [(-\lambda)^1(-1)^0 + (-\lambda)^0(-1)^0] \\ &= (-1)(1-\lambda). \end{aligned}$$

Also, if $|E| = 1$, then $\ell = 1$. Thus, the base case holds.

Inductive Step: Suppose $k \geq 1$. Assume that for single-rooted trees H with $|E(H)| \leq k$, $p(H;\lambda) = (-1)^{|E|}(1-\lambda)^{\ell}$. Now, suppose H is a single-rooted tree with $|E(H)| = k+1$. Consider two cases:

(1) Suppose $\deg_H(r) > 1$

Since $\deg_H(r) > 1$, there exist two subgraphs H_1 and H_2 of H with $H = H_1 \oplus H_2$. Then by Prop. 3, $p(H;\lambda) = p(H_1;\lambda)\cdot p(H_2;\lambda)$. Also, for L_i, the sets of *leaves* of H_i, $L_i = \{v \in V(H_i) - r : \deg_{H_i}(v) = 1\}$, and $\ell_i = |L_i|$, it follows $\ell = \ell_1 + \ell_2$ because E_1 and E_2 form a partition on E and thus $L_1 \bigcap L_2 = \emptyset$. Evaluating $p(H_1;\lambda)\cdot p(H_2;\lambda)$ using the inductive hypotheses yields:

$$\begin{aligned} p(H_1;\lambda)\cdot p(H_2;\lambda) &= (-1)^{|E_1|}(1-\lambda)^{\ell_1}(-1)^{|E_2|}(1-\lambda)^{\ell_2} \\ &= (-1)^{|E_1|+|E_2|}(1-\lambda)^{\ell_1+\ell_2} \\ &= (-1)^{|E|}(1-\lambda)^{\ell}. \end{aligned}$$

Thus, $p(H:\lambda) = (-1)^{|E|}(1-\lambda)^{\ell}$ if $\deg_H(r) > 1$.

(2) Suppose $\deg_H(r) = 1$.

Defined recursively as introduced previously,

$$p(H;\lambda) = (\lambda^{r_H(E)-r_H(E-e)}) \cdot (p(H-e;\lambda)) - p(H/e;\lambda),$$

for the unique $e \in E$, an edge incident to the root. The deletion of e will always lead to the formation of a g-loop, because $\deg_H(r) = 1$ and $|E| = k+1 > 1$. Thus, $p(H-e;\lambda) = 0$, because, according to Gordon and McMahon [2], a graph with a g-loop yields a polynomial of 0. The remaining edges will be unreachable — not connected to the root — since the only edge incident to the root was deleted. The polynomial now equals:

$$p(H;\lambda) = -p(H/e;\lambda).$$

Invoking the inductive hypothesis yields

$$p(G;\lambda) = (-1)(-1)^k(1-\lambda)^\ell = (-1)^{(k+1)}(1-\lambda)^\ell = (-1)^{|E|}(1-\lambda)^\ell.$$

Thus, by the Principle of Mathematical Induction, the polynomial of a single-rooted tree G is given by $p(G;\lambda) = (-1)^{|E|}(1-\lambda)^\ell$.

□

2. Multi-Rooted, Directed Trees

Now, we consider multi-rooted, directed trees. In order to derive a general formula for the characteristic polynomial of multi-rooted, directed trees, we start with two theorems that allow us to consider trees with simpler structures. The next theorem pertains to trees with sinks, i.e. vertices with *out-degree zero*.

Definition 3. The *out-degree* of a vertex v, denoted $deg_-(v)$, is the number of edges emanating and directed outward from this v. Conversely, the *in-degree* of a vertex v, denoted $deg_+(v)$, is the number of edges directed in toward v.

We will see how the existence of a sink allows one to divide the tree at the sink and consider the polynomial as the product of the polynomials of the pieces. The next theorems pertain to a particular structural element we will call a *star*, and we will see that every multi-rooted, directed tree is composed entirely of stars. Finally, we will use all this information to derive a general formula for the characteristic polynomial.

Let $T = (V, E)$ be a multi-rooted, directed tree with vertex set $V(T)$, edge set $E(T)$. Each directed edge in T is associated to an ordered pair of vertices. Let $s \in V$ be a *sink* that is not a leaf, where a *sink* is defined as a vertex with out-degree zero. Note $T - \{s\}$ is not connected, because T is a tree, unless $deg_+(s) = 1$, in which case s would be a leaf as well as a sink. Let $C_1, C_2, \ldots, C_k$ be the components of $T - \{s\}$. Define $V_i = V(C_i) \cup \{s\}$ Let T_i be the graph induced by the vertex set V_i, and let $E_i = E(T_i)$. Then, by definition, $E = \bigcup_{i=1}^{k} E_i$. Also, for all $i \neq j$, $V_i \cap V_j = \{s\}$.

Definition 4. We define a *g-loop* in a directed graph to be an element

(1) In a component without a root.

or

(2) Without a directed path from any $r \in R$ to this element.

Unless otherwise stated, all graphs in this paper will be considered g-loop free, since the existence of a g-loop causes the polynomial to be 0.

Define $\mathcal{F}$ to be the collection of *feasible* subsets $\mathcal{F}(T)$, where a *feasible* subset $F \in \mathcal{F}(T)$ is an acyclic union of single-rooted *rooted arborescences.*

Definition 5. As defined by Gordon and McMahon [2], a rooted subdigraph R is considered a *rooted arborescence* if it contains a root vertex and there exists a unique directed path in R from the root to each vertex in $V(R)$.

For more information, see [4]

Let $\mathcal{F}_i = \mathcal{F}(T_i)$ for $i \in \{1, \ldots, k\}$. In a standard abuse of notation, we let F_i denote both an acyclic union of rooted arborescences and its edge set.

The collection of acyclic unions of rooted arborescences in a multi-rooted, directed tree forms the feasible sets of a *greedoid*.

Definition 6. Let $G = (E, \mathcal{F})$ be a *greedoid* of edge set E and a non-empty collection $\mathcal{F}$ of subsets of edges F with

(1) For each non-empty $F \in \mathcal{F}$ there exists an $e \in F$ such that $F - e \in \mathcal{F}$.
(2) If $F_1, F_2 \in \mathcal{F}$ and $|F_1| > |F_2|$ then there exists an $e \in F_1 - F_2$ with $F_2 \cup e \in \mathcal{F}$.

The sets $F \in \mathcal{F}$ are the feasible sets of G. In a greedoid, an edge is a *g-loop*, or *greedoid loop*, if it is not in any feasible subset. Let $S \subseteq E$; the rank of S, denoted $r_G(S)$ is defined by:

$$r_G(S) = \max\{|F| : F \in \mathcal{F}; F \subseteq S\}.$$

With this new definition of rank, Propositions 1, 2 define the characteristic polynomial for multi-rooted, directed trees. Given two greedoids, $G_1 = (E_1, \mathcal{F}_1)$ and $G_2 = (E_2, \mathcal{F}_2)$, the direct sum is defined as follows:

Definition 7 (Direct Sum of Greedoids)**.** G_1, G_2 comprise the direct sum of G, denoted $G = G_1 \oplus G_2$, if

(1) $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$
(2) $\mathcal{F}(G_1 \oplus G_2) = \{F_1 \cup F_2 | F_1 \in \mathcal{F}(G_1), F_2 \in \mathcal{F}(G_2)\}$

Proposition 4 (Direct Sum for Greedoids; Prop. 3.7 [2])**.** If $G = G_1 \oplus G_2$, then $p(G) = p(G_1) \cdot p(G_2)$.

Theorem 2. *Given a sink s, T and T_i and $G(T), G(T_i)$ for $i \in \{1, ..., k\}$ as defined above,*

$$G(T) = \bigoplus_{i=1}^{k} G(T_i).$$

Proof. In order to prove Theorem 2, it is sufficient to prove $F \in \mathcal{F}$ if and only if $F = \bigcup_{i=1}^{k} F_i$ where $F_i \in \mathcal{F}_i$.

We will prove this in two parts:

(1) Assume $F \in \mathcal{F}$ is a feasible subset of T and let $F_i = F \cap E_i$. It is sufficient to prove for $1 \leq i \leq k$, $F_i \in \mathcal{F}_i$.

For a contradiction, suppose there exists j such that $F_j \notin \mathcal{F}_j$. F_j is not feasible, so there exists $v \in V(F)$ such that there is no unique directed path from a root in F to v, and F is not a union of rooted arborescences. However, the out-degree of the sink s is zero, thus, there exists no unique directed path from any root vertex in $V(F - F_j)$ to this v. Thus, $F \notin \mathcal{F}$ contradicts the assumption that $F \in \mathcal{F}$. Therefore, $F_i \in \mathcal{F}_i$ for all i.

(2) Let $F_i \in \mathcal{F}_i$ for $i \in (1, \ldots, k)$. Now, consider $\bigcup_{i=1}^{k} F_i$. By definition, F_i is an acyclic union of rooted arborescences; thus, $\bigcup_{i=1}^{k} F_i$ is also a union of acyclic rooted arborescences. Since T is a tree, it contains no cycles, and therefore the elements of $\mathcal{F}$ are acyclic unions of rooted arborescences.

Therefore, $\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{F}_i$. Thus, we can now apply Definition 7 to obtain

$$G(T) = \bigoplus_{i=1}^{k} G(T_i).$$

□

Corollary 1 (to Theorem 2)**.** Given T and T_i for $i \in \{1, ..., k\}$ as defined above, by Theorem 2 and the Direct Sum Property in Proposition 4,

$$p(T; \lambda) = \prod_{i=1}^{k} p(T_i; \lambda).$$

We are one step closer to deriving a general formula for the characteristic polynomial of multi-rooted, directed trees. Implicit in Theorem 2 and Corollary 1 is the ability to split graphs at sinks (with in-degree > 1) and form new subgraphs in which the original sink is a leaf.

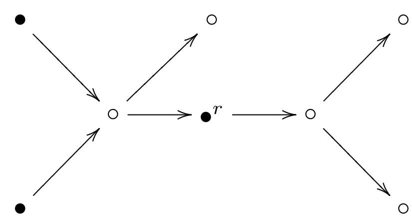


FIGURE 1. G with an internal root

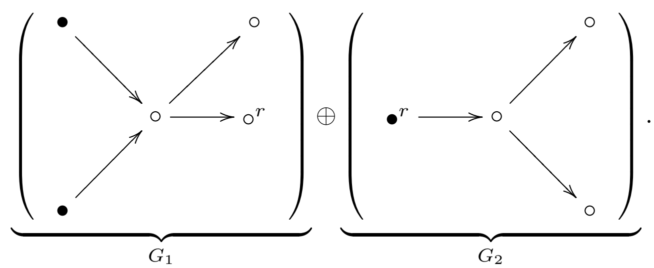


FIGURE 2. $G_1, G_2 \subseteq G$

Now, consider a multi-rooted tree G, and let r be a root with in-degree > 0, which we will call an *internal root*. Let G_1 and G_2 be subgraphs of G such that $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \{r\}$. Let $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(G)$. Let $E(G_2)$ contain all edges connected to those emanating from r, heeding direction. Thus, $E(G_1)$ contains all the edges of $E(G) - E(G_2)$. Let r be a root in G_2 and a non-root, leaf in G_1. See Figure 2 for an example. Figure 2 illustrates the division of G into subgraphs G_1 and G_2.

Lemma 2 (Internal Roots)**.** *Given G, G_1, G_2 and r, an internal root,*

$$G = G_1 \oplus G_2$$

.

Proof. It is sufficient to prove $F \in \mathcal{F}$ if and only if $F = F_1 \cup F_2$ with $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$.

Choose $F \in \mathcal{F}$. F consists of rooted arborescences $A_1, A_2, \ldots, A_k$ such that either $A_i \subseteq G_1$, $A_i \subseteq G_2$, or $A_i \cap G_1 \neq \emptyset$ and $A_i \cap G_2 \neq \emptyset$.
If $A_i \subseteq G_1$, then A_i is clearly a rooted arborescence in G_1. Thus, $F_1 \in \mathcal{F}_1$.
Similarly, if $A_i \subseteq G_2$ then A_i is a rooted arborescence in G_2 and $F_2 \in \mathcal{F}_2$.
If $A_i \cap G_1, G_2 \neq \emptyset$, $A_i \cap G_i$, denoted A_i', is a rooted arborescence in G_1. $A_i \cap G_2$, denoted A_i'' is a rooted arborescence in G_2 since each directed path in A_i to an edge in G_2 passes through r. Then, $A_i = A_i' \cup A_i''$ and $F = F_1 \cup F_2$ with $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$

We can now apply Proposition 4 to obtain $G = G_1 \oplus G_2$.

□

Corollary 2. Given G, G_1, G_2, by Lemma 2 and Definition 7,

$$p(G;\lambda) = p(G_1;\lambda) \cdot p(G_2;\lambda).$$

This allows us to split a graph G with an internal root, r, into *subgraphs* at r. Thus, from now on we will refer to roots as having in-degree 0 unless specified otherwise. We illustrate an example of Theorem 4 in Figure 3:

The next theorem concerns a structure we call a star.

Definition 8. Let $S_{m,n}$ be a directed graph, a *star*, with $m+n+1$ vertices, m roots, n sinks, and 1 vertex with in-degree m, out-degree n.

Theorem 3. *For $S_{m,n}$ as defined above,*

$$p(S_{m,n};\lambda) = (-1)^{m+n}(1-\lambda)^n[(1-\lambda)^m - (-\lambda)^m]. \tag{1}$$

Proof. We will prove this theorem inductively on m. The contraction/deletion formula for polynomials of rooted trees yields the following recursive formula for

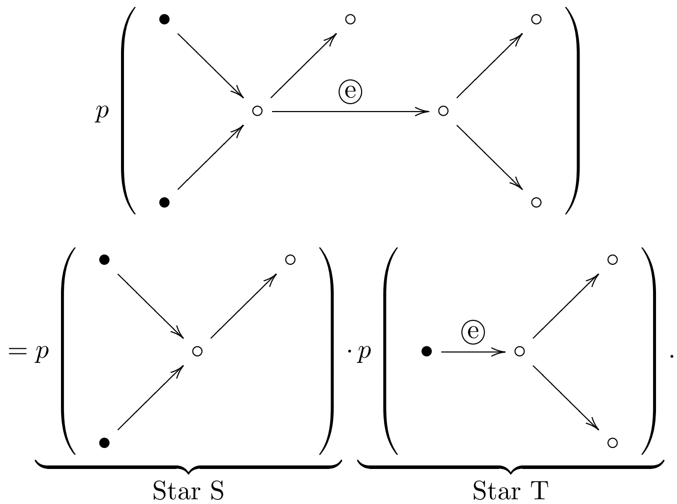


FIGURE 3. Decomposition of S ⓔ T

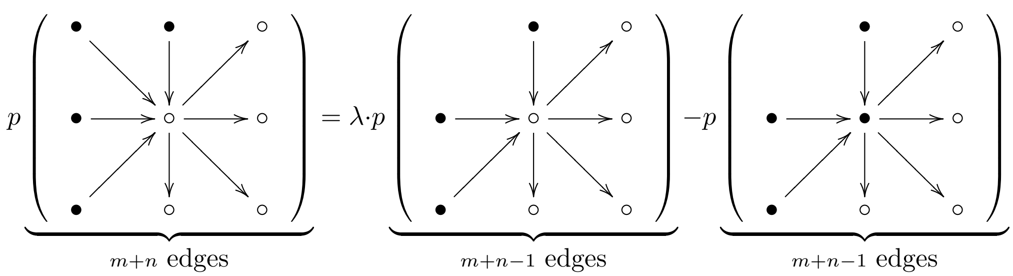


FIGURE 4. Recursive Example

the polynomial of $S_{m,n}$:

$$p(S_{m,n};\lambda) = \lambda p(S_{m-1,n};\lambda) - (-1)^{m+n-1}(1-\lambda)^{m+n-1}.$$

In Figure 4, we illustrate, with an example, the polynomial of a star after one step of contraction/deletion.

We must show (1) complies with the recursive definition for all values of m.

Base Case:

Check to ensure the recursive definition holds for $m = 1$.

$$\begin{aligned} p(S_{1,n};\lambda) &= (-1)^{n+1}(1-\lambda)^n[(1-\lambda)^1 - (-\lambda)^1] \\ &= (-1)^{n+1}(1-\lambda)^n. \end{aligned}$$

Thus, the recursive definition holds for $m = 1$.

Inductive Step:

Now, let $Z_m = p(S_{m,n};\lambda)$ We know (1) holds for Z_{m-1}. Now, we will show it holds for Z_m. Evaluate the right side of the recursive definition as follows:

$$\begin{aligned} Z_m &= \lambda(Z_{m-1}) - (-1)^{m+n-1}(1-\lambda)^{m+n-1} \\ &= \lambda[(-1)^{m+n-1}(1-\lambda)^n[(1-\lambda)^{m-1} - (-\lambda)^{m-1}]] - (-1)^{m+n-1}(1-\lambda)^{m+n-1} \\ &= (-1)^{m+n-1}(1-\lambda)^n[\lambda(1-\lambda)^{m-1} - \lambda(-\lambda)^{m-1} - (1-\lambda)^{m-1})] \\ &= (-1)^{m+n}(1-\lambda)^n[-\lambda(1-\lambda)^{m-1} + \lambda(-\lambda)^{m-1} + (1-\lambda)^{m-1})] \\ &= (-1)^{m+n}(1-\lambda)^n[-\lambda(1-\lambda)^{m-1} - (-\lambda)^m + (1-\lambda)^{m-1})] \\ &= (-1)^{m+n}(1-\lambda)^n[(1-\lambda)^{m-1}(1-\lambda) - (-\lambda)^m] \\ &= (-1)^{m+n}(1-\lambda)^n[(1-\lambda)^m - (-\lambda)^m]. \end{aligned}$$

Thus, by the Principle of Mathematical Induction,

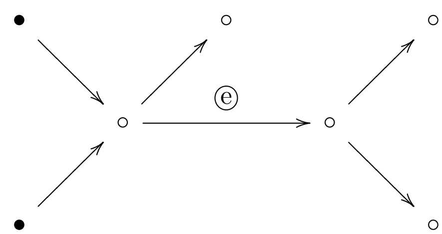
$$p(S_{m,n};\lambda) = (-1)^{m+n}(1-\lambda)^n[(1-\lambda)^m - (-\lambda)^m].$$

□

Theorem 3 leaves us equipped to derive the polynomial of star structures present within multi-rooted directed graphs, and Theorem 2 allows us to split trees at sinks and consider subgraphs of the tree with the sinks as leaves.

We continue with another theorem pertaining to stars. In an effort to view a multi-rooted, directed tree in an even more simplified manner, we consider a tree as being composed solely of stars. The following two theorems confirm the plausibility and usefulness of this point-of-view. The first theorem pertains to the polynomials of stars with shared edges.

Let S, T be stars and S ⓔ T denote the graph containing S, T with one, directed edge from S to T. We consider this edge *shared*. Also, S_{m_0,n_0} ⓔ $(S_{m_1,n_1}, \ldots, S_{m_i,n_i})$ shall denote a star (S_{m_0,n_0}) that shares edges with i other stars.



Next we prove a basic formula for determining the polynomials of stars with shared edges. It is necessary to show a single star, $S_{m,n}$, can always be removed from a multi-rooted tree, T, and the resulting polynomial can be calculated as a product of the polynomial of the removed star and the polynomial of the rest of the tree, T', where $T' = T - S_{m,n}$ and in T', the vertices from which the shared edges emanate become roots. T' may be a forest (collection of trees) or simply a tree. This will allow us to consider the polynomial of a multi-rooted, directed tree entirely in terms of stars. We prove this in the following theorem:

Theorem 4. *Given a multi-rooted tree, T, there exists a star, $S_{m,n}$, an* origin star*, such that if $T' = T - S_{m,n}$ then*

(1) *There exist i directed edges from $S_{m,n}$ to stars in T'.*
(2) $p(T) = p(S_{m,n-i}) \cdot p(T')$.

Proof. First, the existence of such a star is guaranteed by the definition of a directed tree. Consider any internal vertex; the adjacent edges are either directed in or out. Since cycles cannot exist, we can continue backwards along edges directed from one internal vertex to another until we eventually reach an internal vertex such that the only adjacent edges directed in are from roots. Therefore, there exists a star from which the tree T originates, the *origin star*.

We will prove this by induction on m.

Base Case: Let $m = 1$.

Evaluating the polynomial using the contraction/deletion formula yields a first term (the deletion term) which is the polynomial of a graph with a g-loop and thus must be zero.

$$\begin{aligned} p(S_{1,n} ⓔ T') &= \lambda(0) - (-1)^{n-i}(1-\lambda)^{n-i} \cdot p(T') \\ &= (-1)^{n-i+1}(1-\lambda)^{n-i} \cdot p(T') \\ &= p(S_{1,n-i}) \cdot p(T'). \end{aligned}$$

Thus, the base case holds.

Inductive Step:

Inductive Hypothesis:

For $m = k$ assume $p(S_{k,n} ⓔ T') = p(S_{k,n-i}) \cdot p(T')$.

Now, let $m = k+1$ and show $p(S_{k+1,n} ⓔ T') = p(S_{k+1,n-i}) \cdot p(T')$.

$$\begin{aligned} & p(S_{k+1,n} ⓔ T') \\ &= \lambda p(S_{k,n} ⓔ T') - (-1)^{k+n-i}(1-\lambda)^{k+n-i} \cdot p(T') \\ &= \lambda p(S_{k,n-i}) \cdot p(T') - (-1)^{k+n-i}(1-\lambda)^{k+n-i} \cdot p(T') \\ &= \lambda(-1)^{k+n-i}(1-\lambda)^{n-i}[(1-\lambda)^k - (-\lambda)^k] \\ &\quad -(-1)^{k+n-i}(1-\lambda)^{k+n-i} \cdot p(T') \\ &= p(T') \cdot (-1)^{k+n-i}(1-\lambda)^{n-i}[\lambda(1-\lambda)^k - \lambda(-\lambda)^k - (1-\lambda)^k] \\ &= p(T') \cdot (-1)^{k+1+n-i}(1-\lambda)^{n-i}[(1-\lambda)^k - \lambda(1-\lambda)^k + \lambda(-\lambda)^k] \\ &= p(T') \cdot (-1)^{k+1+n-i}(1-\lambda)^{n-i}[(1-\lambda)^k(1-\lambda) - (-\lambda)^{k+1}] \\ &= p(T') \cdot (-1)^{k+1+n-i}(1-\lambda)^{n-i}[(1-\lambda)^{k+1} - (-\lambda)^{k+1}] \\ &= p(S_{k+1,n-i}) \cdot p(T'). \end{aligned}$$

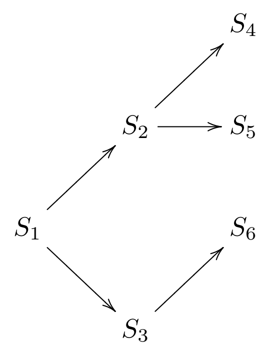


FIGURE 5. A "Cluster" of Stars

Therefore, by the Principle of Mathematical Induction,

$$p(T) = p(S_{m,n-i}) \cdot p(T').$$

□

Corollary 3 (to Theorem 4)**.** In a multi-rooted, directed tree T there exist $S_1, S_2, \ldots, S_k$ stars. In S_i, define m_i as edges directed in towards S_i and n_i as edges directed out from S_i excluding shared edges. Then, by Theorems 3 and 4,

$$p(T) = \prod_{i=1}^{k} p(S_i) = \prod_{i=1}^{k} \left[(-1)^{m_i+n_i}(1-\lambda)^{n_i}[(1-\lambda)^{m_i} - (-\lambda)^{m_i}]\right].$$

These proofs tells us that each multi-rooted, directed tree can be considered as a cluster of stars, and the characteristic polynomial can be evaluated in terms of the stars. In an effort to develop a general formula for the characteristic polynomial of these trees, I note that these structures can be classified according to their vertices and that there are exactly three possible types of vertices: *sinks*, *roots*, and *internal vertices*. Our final theorem provides us with a general formula for the characteristic polynomial composed of each type of vertex's specific contribution.

Let $I \subseteq V$ be the set of internal vertices in V, and let $S \subseteq V$ be the set of sinks in V. Before we prove the final theorem, we first prove a pertinent lemma.

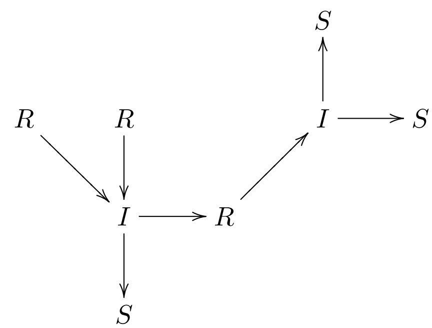


FIGURE 6. Edge Destinations

Lemma 3.

$$|V| - 1 = \sum_{s \in S} deg_+(s) + \sum_{r \in R} deg_+(r) + \sum_{v \in I} deg_+(v).$$

Proof. We know $|V| - 1 = |E|$. Every $e \in E$ has a source and a destination. The destination of an edge can either be an internal vertex, a sink, or a root. Thus, it is either a member of the in-degree of an internal vertex, a sink, or a root. The in-degree of a vertex reflects the number of edges for which it serves as a destination. Thus, the total number of edges ending in a sink is equal to $\sum_{s \in S} deg_+(s)$. Similarly, the total number of edges ending in an internal vertex is equal to $\sum_{v \in I} deg_+(v)$. Also, the total number of edges ending in a root is equal to $\sum_{r \in R} deg_+(r)$. Since there are only three types of vertices, it follows that:

$$|V| - 1 = \sum_{s \in S} deg_+(s) + \sum_{r \in R} deg_+(r) + \sum_{v \in I} deg_+(v).$$

Figure 6 illustrates the proof of this lemma through example. In the figure, R denotes a root, I an internal vertex, and S a sink.

□

Theorem 5. *Consider a multi-rooted, directed tree, T. Then, the polynomial of T is defined:*

$$p(T) = (-1)^{|V|-1}(1-\lambda)^{\sum_{s\in S} deg_+(s)+\sum_{r\in R} deg_+(r)} \prod_{v\in I} \left[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}\right].$$

Proof. In order to prove this theorem, we consider the three specific types of vertices. We will show that the vertices contribute the following terms to a product which comprises the characteristic polynomial as proposed:

Sink: $(-1)^{deg_+(s)}(1-\lambda)^{deg_+(s)}$

Internal Vertex: $(-1)^{deg_+(v)}[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}]$

Root: $(-1)^{deg_+(r)}(1-\lambda)^{deg_+(r)}$

Therefore, roots with in-degree equal to zero contribute trivially to the polynomial (a factor of one). Since we have shown it is possible to consider a multi-rooted, directed tree as a cluster of stars, our proof here will utilize this. We show now that the basic polynomial for a star $S_{m,n}$, with in-degree m, out-degree n, as defined in Theorem 3, satisfies this theorem. In a single star, the only roots are those with in-degree 0; thus the out-degree n equals $|S|$. Every sink is a leaf, so the in-degree of each sink is 1. Thus, $n = |S| = \sum_{s\in S} deg_+(s)$. Consider

$$\begin{aligned} & p(S_{m,n};\lambda) \\ &= (-1)^{m+n}(1-\lambda)^n[(1-\lambda)^m - (-\lambda)^m] \\ &= \underbrace{[(-1)^n(1-\lambda)^n]}_{\substack{n \text{ sinks} \\ \text{with in-degree 1}}} \cdot \underbrace{[(-1)^m[(1-\lambda)^m - (-\lambda)^m]]}_{\substack{\text{one internal vertex} \\ \text{with in-degree } m}} \cdot \underbrace{[(-1)^0(1-\lambda)^0]}_{\substack{0 \text{ roots} \\ \text{with in-degree} > 0}}. \end{aligned}$$

Thus, every star can be expressed as a product of terms determined by the in-degree associated with the types of vertices that comprise it.

Theorem 4 shows every multi-rooted, directed tree is composed of connected stars, and the corresponding polynomial of every tree can be treated as a product of terms derived from the stars. Now, we have shown it is possible to express a star as a product of terms derived from the specific vertices. Thus, the polynomial of all multi-rooted, directed trees can be expressed in terms of the specific vertex contributions as listed. Consider:

$$
\begin{aligned}
p(T) & \\
&= (-1)^{|V|-1}(1-\lambda)^{\sum_{s\in S} deg_+(s)+\sum_{r\in R} deg_+(r)} \prod_{v\in I}\left[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}\right] \\
&= (-1)^{\sum_{s\in S} deg_+(s)+\sum_{v\in I} deg_+(v)+\sum_{r\in R} deg_+(r)}(1-\lambda)^{\sum_{s\in S} deg_+(s)+\sum_{r\in R} deg_+(r)} \\
&\quad \cdot \prod_{v\in I}\left[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}\right] \\
&= (-1)^{\sum_{s\in S} deg_+(s)}(1-\lambda)^{\sum_{s\in S} deg_+(s)}(-1)^{\sum_{r\in R} deg_+(r)}(1-\lambda)^{\sum_{r\in R} deg_+(r)} \\
&\quad \cdot (-1)^{\sum_{v\in I} deg_+(v)} \prod_{v\in I}\left[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}\right] \\
&= \underbrace{(-1)^{\sum_{s\in S} deg_+(s)}(1-\lambda)^{\sum_{s\in S} deg_+(s)}}_{|S| \text{ sinks}} \underbrace{(-1)^{\sum_{r\in R} deg_+(r)}(1-\lambda)^{\sum_{r\in R} deg_+(r)}}_{\substack{|R| \text{ roots} \\ \text{with in-degree} > 0}} \\
&\quad \underbrace{\cdot \prod_{v\in I}\left[(-1)^{deg_+(v)}[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}]\right].}_{|I| \text{ internal vertices}}
\end{aligned}
$$

Thus, the characteristic polynomial of a multi-rooted, directed tree T is:

$$p(T) = (-1)^{|V|-1}(1-\lambda)^{\sum_{s\in S} deg_+(s)+\sum_{r\in R} deg_+(r)} \prod_{v\in I}\left[(1-\lambda)^{deg_+(v)} - (-\lambda)^{deg_+(v)}\right].$$

□

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