A Dimension Formula for Modular Forms
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The goal of this paper is to establish a dimension formula for two classes of modular forms on the full modular group. Modular Forms are a class of analytic functions on the upper half of the complex plane. These functions are of great interest in number theory research due to their symmetry properties. We will first discuss the definition of a modular form and the terminology necessary to understand the definition. We will then define two specific types of modular forms, entire forms and cusp forms before discussing a set of entire forms, the Eisenstein Series and a cusp form, the Discriminant function. Finally we will use these two types of functions to establish a dimension formula for the vector spaces of entire and cusp modular forms on the full modular group.

We will use the following notation throughout the paper:

- \( \mathbb{Z} \) is the set of integers,
- \( \mathbb{R} \) denotes the real line,
- \( \mathbb{C} \) is the complex plane,
- \( \mathbb{S} = \mathbb{C} \cup \{\infty\} \) is the Riemann sphere, and
- \( \mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\} \) is the upper half plane.

1 Introduction

Over the course of the Section 1 we will establish the definition of a Modular Form, first stating the definition in Section 1.1. Then in Sections 1.2 through 1.6 we will define aspects of the definition of a modular form that may be unfamiliar to the reader. Terms that are defined in sections after the one in which they are first presented are underlined. Once the definition of a modular form has been established in Section 2 we will establish some of the properties and notation used for modular forms before finally looking at some specific modular forms: in Section 3 the Eisenstein Series, and the Discriminant Function in Section 4. Finally in Section 5 we will discuss what we are able to learn about the vector spaces of modular forms from these functions.

1.1 Definition

A modular form is defined by the following. Suppose that \( \Gamma \) is a subgroup of finite index in \( \Gamma(1) \) with standard fundamental region \( \mathcal{R} \), \( k \in \mathbb{R} \), \( v \) is a multiplier system for \( \Gamma \) with weight \( k \), and that \( F(z) \) is a function meromorphic in \( \mathcal{H} \). Then \( F(z) \) is a modular form of weight \( k \) with multiplier system \( v \) with respect to \( \Gamma \) if \( F \) satisfies the transformation law

\[
F(M\tau) = v(M)(c\tau + d)^k F(\tau)
\]

for every \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \), \( F \) has at most a finite number of poles in \( \mathcal{R} \cap \mathcal{H} \), and \( F \) is meromorphic at each parabolic cusp of \( \Gamma \in \mathbb{R} \).

This definition will be restated in Section 1.6.
1.2 $Γ(1)$

Modular forms are defined on subgroups $Γ$ of $Γ(1)$, or the modular group. The modular group is defined to be the set of linear fractional transformations with integer coefficients and determinant 1. Linear fractional transformations are functions of the form

$$Mz = \frac{az + b}{cz + d}$$

where $a, b, c, d$ are complex numbers and $z \in S$. $S = C \cup \{\infty\}$ is called the Riemann sphere. In the Riemann sphere $\infty$ is considered to be a real number. So $Γ(1) = \{Mz = \frac{az + b}{cz + d} | a, b, c, d \in Z, ad - bc = 1\}$. This set of functions is a group under the operation function composition and the identity $I(z) = \frac{1 + 0}{0z + 1} = z$.

**Lemma 1.1.** $Γ(1)$ is closed under the operation of function composition.

**Proof.** Let $M_1z = \frac{a_1z + b_1}{c_1z + d_1}$ and $M_2z = \frac{a_2z + b_2}{c_2z + d_2}$.

$$M_1(M_2(z)) = M_1\left(\frac{a_2z + b_2}{c_2z + d_2}\right)$$

$$= \frac{a_1\frac{a_2z + b_2}{c_2z + d_2} + b_1}{c_1\frac{a_2z + b_2}{c_2z + d_2} + d_1}$$

$$= \frac{a_1\frac{a_2z + b_2}{c_2z + d_2} + b_1}{c_1\frac{a_2z + b_2}{c_2z + d_2} + d_1} \cdot \frac{c_2z + d_2}{c_2z + d_1}$$

$$= \frac{a_1(a_2z + b_2) + b_1(c_2z + d_2)}{c_1(a_2z + b_2) + d_1(c_2z + d_2)}$$

$$= \frac{(a_1a_2 + b_1c_2)z + a_1b_2 + b_1d_2}{(c_1a_2 + d_1c_2)z + c_1b_2 + d_1d_2}$$

We must check that this satisfies the condition $ad - bc = 1$.

$$(a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)(c_1a_2 + d_1c_2) = a_1a_2c_1b_2 + a_1a_2d_1d_2 + b_1c_2b_1c_2 + b_1c_2d_1d_2$$

$$- a_1b_2c_1a_2 - a_1b_2d_1c_2 - b_1d_2c_2d_2 - b_1d_2a_1c_2 - b_1d_2a_1d_1$$

$$= a_2d_2(a_1d_1 - b_1c_1) + b_2c_2(b_1c_2 - a_1d_1).$$

Recall that $a_1d_1 - b_1c_1 = 1$ and $a_2d_2 - b_2c_2 = 1$, so

$$a_2d_2(a_1d_1 - b_1c_1) + b_2c_2(b_1c_2 - a_1d_1) = a_2d_2 - b_2c_2 = 1.$$ 

Therefore $Γ(1)$ is closed under function composition.

Consider the following Lemma

**Lemma 1.2.** All real linear fractional transformations with non-zero determinants can be normalized such that for $Mz = \frac{az + b}{cz + d}$, $ad - bc = 1$.

**Proof.** First consider a linear fractional transformation where $ad - bc = x$. If we multiply by $1 = \frac{x}{x}$ to give $\frac{ma + mb}{mcz + md}$ we will end up with $a'd' - b'c' = mad - mbc = m^2(ad - bc) = m^2x$.

If $m^2 = \frac{1}{x}$, or $m = \frac{1}{\sqrt{ad - bc}}$, then $a'd' - b'c' = 1$. 

\[\square]\
Linear fractional transformations are often identified with matrices where the linear fractional transformation $Mz = \frac{az + b}{cz + d}$ is represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consequently $ad - bc$ is called the determinant of the linear fractional transformation, and the matrices $M$ and $-M$ are equivalent. The matrices that represent linear fractional transformations are the special linear group, $SL(2, \mathbb{Z})$, which is the set of two-by-two matrices with integer coefficients and determinant 1. There exists a map from $SL(2, \mathbb{Z})$ to $\Gamma(1)$ that is a homomorphism with kernel $\pm I$.

**Lemma 1.3.** Matrix multiplication in $SL(2, \mathbb{Z})$ is equivalent to function composition in $\Gamma$.

**Proof.** First look at matrix multiplication:

$$M_1 M_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$ 

When we write this matrix as a linear fractional transformation we get that

$$M_1 M_2(z) = \frac{(a_1 a_2 + b_1 c_2)z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2}.$$ 

Recall that above we calculated that function composition gives the same result,

$$M_1(M_2(z)) = \frac{(a_1 a_2 + b_1 c_2)z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2}.$$ 

\[\square\]

Real linear fractional transformations of determinant 1 are classified into three categories. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear fractional transformation of determinant 1, then $M$ is either elliptic if $|a + d| < 2$, hyperbolic if $|a + d| > 2$, or parabolic if $|a + d| = 2$. These classifications can also be made by observing the number of fixed points that a linear fractional transformation has, as in the following lemma.

**Lemma 1.4.** Suppose $\Gamma$ is a group of real linear fractional transformations of determinant 1, and suppose that $M \in \Gamma$. Then

a) $M$ is elliptic if and only if $M$ has two complex conjugate fixed points;

b) $M$ is hyperbolic if and only if $M$ has two distinct real fixed points; and

c) $M$ is parabolic if and only if $M$ has one real fixed point.

**Proof.** A point $z$ is a fixed point of a linear fractional transformation $M$ if $Mz = z$. Then
\[
\frac{az+b}{cz+d} = z \quad \text{and} \quad az + b = cz^2 + dz \quad \text{and} \quad cz^2 + (d-a)z - b = 0. \quad \text{So by the quadratic formula,}
\]
\[
z = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4cb}}{2c} = \frac{(a-d) \pm \sqrt{a^2 - 2ad + d^2 + 4cb}}{2c} = \frac{(a-d) \pm \sqrt{a^2 + d^2 + 2ad - 4ad + 4cb}}{2c} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}.
\]

Now consider the term \(\sqrt{(a+d)^2 - 4}\). If \(M\) is elliptic, then \(|a+d| < 2\) and \((a+d)^2 - 4\) is negative so there will be two complex values for \(z\). If \(M\) is hyperbolic, then \(|a+d| > 2\) and \((a+d)^2 - 4\) is positive so there will be two distinct real values \(z\). If \(M\) is parabolic, then \(|a+d| = 2\) so \((a+d)^2 - 4 = 0\) so there will be one real value for \(z\). \(\square\)

### 1.2.1 Important Matrices in \(\Gamma(1)\)

The following matrices in \(\Gamma(1)\) will be important to the remainder of the paper. The matrix
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
is the identity. The matrix
\[
S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
represents the linear fractional transformation that is translation by 1, and the matrix
\[
T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
represents the linear fractional transformation that is \(-\frac{1}{z}\). Note that both \(T^2 = I\), and \((ST)^3 = I\). These calculations are shown in Appendix A. Additionally for every \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\), \(M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\). One subgroup of \(\Gamma(1)\) that we will discuss, the theta group \(\Gamma_\theta\), is the group generated by \(S^2\) and \(T\).

### 1.3 Fundamental Region

A fundamental region is defined as follows. Suppose \(\Gamma\) is a discrete group of real linear fractional transformations. A fundamental region for \(\Gamma\) in \(\mathcal{H}\) is an open set \(R \subseteq \mathcal{H}\) such that no two distinct points of \(R\) are equivalent with respect to \(\Gamma\), and every point of \(\mathcal{H}\) is equivalent to some point of \(R\) with respect to \(\Gamma\). Discrete groups, equivalence and standard fundamental region are defined in the subsections below. Note that a fundamental region is not necessarily connected, so it may not be a region, and that fundamental regions are not unique. Due to the fact that every point in the upper half plane is equivalent to a point in the fundamental region if we can understand what happens to the fundamental region, we can understand what happens to the entire upper half plane. This makes fundamental regions very important to the study of modular forms.

### 1.3.1 Discrete Group

A group is discrete if it contains no convergent sequence of distinct elements. Closely related to the idea of discrete groups are limit points and discontinuity so we will discuss them here. An \(\alpha \in S\) is called a limit point of the group of real linear fractional transformations \(\Gamma\) if
there exists \( z \in S \) and a sequence \( \{V_n\} \) of distinct elements of \( \Gamma \) such that \( V_nz \to \alpha \). The limit set \( L = L(\Gamma) \) is the set of all limit points of \( \Gamma \). If a point is not a limit point it is called an ordinary point. The set of ordinary points is denoted \( O(\Gamma) = \emptyset \). A group \( \Gamma \) is called discontinuous if \( O(\Gamma) \neq \emptyset \). One useful property of groups of real linear fractional transformations is that they are discontinuous if and only if they are discrete.

A consequence of the proof of this fact is that if \( \Gamma \) is discrete, then \( L(\Gamma) \subseteq \mathbb{R} \). If \( \Gamma \) is not discrete, then \( L(\Gamma) = S \). Since \( \Gamma \) is a group of real linear fractional transformations, if it is not discrete, it is not discontinuous, and so \( L = S \).

Additionally if \( \Gamma \) is an infinite group of linear fractional transformations, \( L(\Gamma) \neq \emptyset \). Since the Riemann sphere is compact, the sequence \( \{V_nz\} \), where \( V_n \in \Gamma \), must have some convergent subsequence. So therefore there exists some \( \alpha \) such that some subsequence of \( \{V_nz\} \) at least one limit point of \( \Gamma \) in \( S \).

1.3.2 Equivalence

Two points \( z_1, z_2 \in S \) are equivalent with respect to a group of linear fractional transformations \( \Gamma \) if there exists \( V \in \Gamma \) such that \( Vz_1 = z_2 \).

1.3.3 Finite Index

A subgroup is of finite index if it has finitely many coset representatives. For example the subgroup \( \Gamma_\theta \) of \( \Gamma(1) \) has cosets \( I\Gamma_\theta \), \( S^{-1}\Gamma_\theta \) and \( S^{-1}T\Gamma_\theta \), where \( S \) and \( T \) are the matricies defined in section 1.2.1. The union of the two non-trivial cosets and \( \Gamma_\theta \) is equal to \( \Gamma(1) \), so we say that \( \Gamma_\theta \) has index three.

1.3.4 Standard Fundamental Region

Suppose \( \Gamma \) is a subgroup of \( \Gamma(1) \) of finite index \( \mu \) with coset representatives \( A_1, A_2, ... A_\mu \), so \( \Gamma(1) = \cup_{i=1}^{\mu} \Gamma A_i \). Then we call \( R = \cup_{i=1}^{\mu} A_i \{R(\Gamma(1))\} \) the standard fundamental region of \( \Gamma \) is . \( R(\Gamma(1)) \) is the standard fundamental region for \( \Gamma(1) \), defined to be \( R(\Gamma(1)) = \{z \in \mathbb{H} | |z| > 1, |Re(z)| < \frac{1}{2} \} \). This is shown in the grey region in Figure 1.

![Figure 1: Standard Fundamental Regions of \( \Gamma(1) \) (grey) and \( \Gamma_\theta \) (red+grey) and the circles and lines used to create other fundamental regions](image)

If we consider \( \Gamma_\theta \) as in Section 1.3.3 we can then find its standard fundamental region using the formula given above. The coset representatives of \( \Gamma_\theta \) are \( I, S^{-1}, \) and \( S^{-1}T \), so the standard fundamental region of \( \Gamma_\theta \) is the union of \( I(R\Gamma(1)), S^{-1}(R(\Gamma(1))) \) and
Expansions. In order to say that a function is meromorphic at a cusp we must first discuss Fourier

1.5.1 Fourier Series

\[ q \Gamma, \] we say the width of \( \Gamma(1) \)

that are chosen.

1.4 Multiplier Systems

A multiplier system is defined as follows; a function \( v : \Gamma \rightarrow \mathbb{C} \) is called a multiplier system of weight \( k \) on the group \( \Gamma \) if \( |v(M)| = 1 \) for all linear fractional transformations \( M \in \Gamma \) and if \( v \) satisfies the consistency condition

\[
v(M_1 M_2)(c_3 \tau + d_3)^k = v(M_1) v(M_2)(c_1 M_2 \tau + d_1)^k(c_2 \tau + d_2)^k
\]

where \( M_1 = \begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix} \), \( M_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix} \), and \( M_1 M_2 = \begin{pmatrix} * & * \\ c_3 & d_3 \end{pmatrix} \) for \( \tau \in \mathcal{H} \).

Note that this condition arises naturally from the transformation law presented for modular forms,

\[ F(M \tau) = v(M)(c \tau + d)^k F(\tau) \]

since

\[
F(M_1 M_2(\tau)) = F(M_1(M_2 \tau)) \\
= v(M_1)(c_1 M_2 \tau + d_1)^k F(M_2 \tau) \\
= v(M_1)(c_1 M_2 \tau + d_1)^k v(M_2)(c_2 \tau + d_2)^k F(\tau)
\]

and

\[ F(M_1 M_2(\tau)) = v(M_1 M_2)(c_3 \tau + d_3)^k F(\tau). \]

Therefore the only way that these equations are both true is if

\[
v(M_1 M_2)(c_3 \tau + d_3)^k = v(M_1) v(M_2)(c_1 M_2 \tau + d_1)^k(c_2 \tau + d_2)^k.
\]

1.5 Parabolic Cusp

A parabolic cusp (or a parabolic point) of a fundamental region \( R \) of \( \Gamma \) is defined to be any point \( q \in \mathbb{R} \) (including infinity), such that \( q \in R \). If we consider \( R(\Gamma(1)) \), the only real number in \( R(\Gamma(1)) \) is \( \infty \) so the only parabolic cusp of \( \Gamma(1) \) is \( \infty \). Now consider a subgroup of \( \Gamma(1) \) with finite index \( \mu \), \( \Gamma \) with coset representatives \( A_1, A_2, \ldots, A_\mu \); and let \( R \) be the standard fundamental region as given in Section 3.3. Then the parabolic cusps of \( \Gamma \) in \( R \) are \( \{A_1 \infty, A_2 \infty, \ldots, A_\mu \infty \} \). Note that some of these representations may give the same points.

Parabolic cusps have a characteristic called width. For \( \Gamma \) a subgroup of finite index of \( \Gamma(1) \), with parabolic points \( q_j = A_j \infty, 1 \leq j \leq \mu \) in the standard fundamental region \( R \) of \( \Gamma \), we say the width of \( q_j \) is the smallest positive integer \( \lambda_j \) such that \( S^{\lambda_j} \in A_j^{-1} \Gamma A_j \).

1.5.1 Fourier Series

In order to say that a function is meromorphic at a cusp we must first discuss Fourier Expansions.
Let \( \Gamma \) be a subgroup of finite index in \( \Gamma(1) \) and suppose that \( v \) is a multiplier system of weight \( k \) for \( \Gamma \). Suppose that \( q = A(\infty) \) for \( A \in \Gamma \) is a parabolic point of \( \Gamma \), that \( M_q = ASA^{-1} \), and that \( \lambda \) is the width of \( q \) as defined in the previous section. We define the real number \( \kappa_q \) by
\[
v(M^\lambda_q) = e^{2\pi i \kappa_q},
\]
with \( 0 \leq \kappa_q < 1 \).

**Theorem 1.1.** [4] Suppose that \( \Gamma \) is a subgroup of finite index in \( \Gamma(1) \) with standard fundamental region \( R \) given above. Suppose that \( F(\tau) \) is meromorphic in \( \mathcal{H} \), satisfies the transformation law,
\[
F(M\tau) = v(M)(c\tau + d)^k F(\tau),
\]
for every \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \), and has at most a finite number of poles in \( \overline{R} \cap \mathcal{H} \). Let \( q_1 = \infty, q_2, \ldots, q_p \) be the inequivalent parabolic points of \( R \), with corresponding widths \( \lambda_1, \lambda_2, \ldots, \lambda_p \) and corresponding numbers \( \kappa_1, \kappa_2, \ldots, \kappa_p \) from the multiplier system \( v \). Then for each \( j, 1 \leq j \leq p \), there exists a nonnegative real number \( y_j \) such that \( F(\tau) \) has the Fourier expansion valid for \( \text{Im}(A^{-1}_j \tau) > y_j \),
\[
F(\tau) = \sigma_j(\tau) \sum_{n = -\infty}^{\infty} a_n(j) e^{2\pi i (n + \kappa_j)(A^{-1}_j(\tau))/\lambda_j}
\]
where \( \sigma_j(\tau) = 1 \) if \( j = 1 \) or \( \sigma_j(\tau) = (\tau - q_j)^{-k} \) for \( 2 \leq j \leq p \).

Note that if \( \Gamma = \Gamma(1) \), there is only one expansion, at infinity. However, if \( \Gamma = \Gamma_{\theta} \) then there are two non-equivalent cusps, one at \(-1\) and one at infinity, and therefore there are two Fourier expansions of modular forms in \( \Gamma_{\theta} \).

### 1.5.2 Meromorphic at a Cusp

Suppose \( F(\tau) \) is a function defined in \( \mathcal{H} \) satisfying the conditions of Theorem 1.1 above. If a finite number of terms with \( n < 0 \) appear in the equation for the Fourier expansion we say \( F(\tau) \) is *meromorphic* at \( q_j \). If \( a_n(j) = 0 \) for all \( n < 0 \), then \( F(\tau) \) is said to be *regular* at \( q_j \).

### 1.6 The Definition of a Modular Form

We will now restate the definition of a modular form before beginning the discussion of some of the properties of modular forms in the next section. Suppose that \( \Gamma \) is a subgroup of finite index in \( \Gamma(1) \) with standard fundamental region \( \overline{R}, k \in \mathbb{R}, \) \( v \) is a multiplier system for \( \Gamma \) with weight \( k \), and that \( F(z) \) is a function meromorphic in \( \mathcal{H} \). Then \( F(z) \) is a modular form of weight \( k \) with multiplier system \( v \) with respect to \( \Gamma \) if \( F \) satisfies the transformation law
\[
F(M\tau) = v(M)(c\tau + d)^k F(\tau)
\]
for every \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \), \( F \) has at most a finite number of poles in \( \overline{R} \cap \mathcal{H} \), and \( F \) is meromorphic at each parabolic cusp of \( \Gamma \in \mathbb{R} \).
2 Properties of Modular Forms

In this section we will discuss properties of different types modular forms, particularly when certain aspects are restricted, such as the number of cusps and the multiplier system. Following this section we will be focusing on some entire modular forms (particularly the Eisenstein Series), and one cusp form (the discriminant function).

2.1 Types of Modular Forms

An entire modular form is a modular form that is regular at every cusp and holomorphic in \( \mathcal{H} \). Recall from section 1.5.2 that this means all of the coefficients for the negative indices of the summation in the Fourier expansion are equal to zero.

A cusp modular form is an entire modular form for which the first term for which \( a_n \neq 0 \) in the Fourier expansion at each cusp is a term such that \( n + \kappa > 0 \). Note that not all entire forms are cusp forms; if the smallest value of \( n \) such that \( a_n \neq 0 \) in the Fourier expansion occurs at \( n = 0 \) and \( \kappa = 0 \) then the first nonzero term does not satisfy the condition for cusp forms. All entire modular forms have a non-negative weight.

Theorem 2.1. If \( F(\tau) \) is an entire or cusp modular form with weight less than zero then \( F(\tau) \equiv 0 \).

Proof. A proof that this is true in \( \Gamma(1) \) is given in Appendix B.

A modular function is a modular form of weight \( k = 0 \) and multiplier system \( v \equiv 1 \).

Theorem 2.2. If a modular function \( f \) is bounded in \( \mathcal{H} \), then \( f \) is constant.

2.2 Vector Spaces of Modular Forms

The following notation will be used to describe the vector spaces of modular forms. \( \{\Gamma(1), k, v\} \) is a vector space with operation function addition with traditional scalar multiplication and the identity \( f(z) = 0 \). In order for subsets of \( \Gamma(1) \) to be subspaces of \( \{\Gamma(1), k, v\} \) they must be closed under function addition and scalar multiplication.

- \( \{\Gamma, k, v\} \) is the vector space over \( \mathbb{C} \) of all modular forms with weight \( k \) and multiplier system \( v \) with respect to \( \Gamma \).
- \( C^+(\Gamma, k, v) \) is the vector space over \( \mathbb{C} \) of all entire modular forms with weight \( k \) and multiplier system \( v \) with respect to \( \Gamma \).
- \( C^0(\Gamma, k, v) \) is the vector space over \( \mathbb{C} \) of all cusp modular forms with weight \( k \) and multiplier system \( v \) with respect to \( \Gamma \).
- If the multiplier system is \( v \equiv 1 \), then we simply write \( \{\Gamma, k\} \), \( C^+(\Gamma, k) \), and \( C^0(\Gamma, k) \).

In Section 5.1 we will establish a formula for the dimension of \( C^0(\Gamma(1), k) \) and \( C^+(\Gamma(1), k) \) when \( k \) is even. Note that these subspaces are not closed under function multiplication; a modular form of weight four times a modular form of weight four has weight eight, and a modular form of weight ten divided by a modular form of weight four has weight six. This fact is explicitly stated in the following lemma;

Lemma 2.1. Let \( F_1 \in \{\Gamma, k_1, v_1\} \) and \( F_2 \in \{\Gamma, k_2, v_2\} \). Then
(i) $F_1 F_2 \in \{ \Gamma, k_1 + k_2, v_1 v_2 \}$, and

(ii) $\frac{F_1}{F_2} \in \{ \Gamma, k_1 - k_2, \frac{v_1}{v_2} \}$ if $F_2 \neq 0$.

A calculation outlining this fact is given in Appendix C.

2.3 Modular Forms on $\Gamma(1)$

If a modular form is defined on the entire modular group, $\Gamma(1)$, then, as discussed above it has one cusp at infinity. This means that there exists only one Fourier expansion for this modular form but it also simplifies the formula. $\sigma_1(\tau) = 1$, so the coefficient disappears. Additionally the coset representative, $A_1 = I$, so $A_1 \tau = \tau$. This leaves us with

$$F(\tau) = \sum_{-\infty}^{\infty} a_n e^{2\pi i (n+\kappa) \tau / \lambda}$$

for $\tau \in \mathcal{H}$.

2.4 The Simplest Case

For the remainder of the paper we will work with an even further simplified case of modular forms. These are modular forms on the entire modular group, so they only have one cusp at infinity, since infinity is the only real number in the standard fundamental region for $\Gamma(1)$. These modular forms also have width $\lambda = 1$, and multiplier system $\nu \equiv 1$, which gives $\kappa = 0$. This leaves us with a Fourier expansion of

$$F(\tau) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n \tau}$$

for $\tau \in \mathcal{H}$.

3 Eisenstein Series

This section discusses the Eisenstein Series. The Eisenstein Series are the entire modular forms on $\Gamma(1)$. The Eisenstein series $G_k(\tau)$ is defined by

$$G_k(\tau) = \sum_{c} \sum_{d} (c \tau + d)^{-k}$$

where $c, d \in \mathbb{Z}$, $\tau \in \mathcal{H}$, and $k > 2$. The $'$ signifies that $c$ and $d$ are not both zero.

The normalized Eisenstein series is given by

$$E_k(\tau) = \frac{1}{\pi} \sum_{c} \sum_{d \ (c,d)=1} (c \tau + d)^{-k}$$

for $\tau \in \mathcal{H}$, and $k > 2$. 

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3.1 Properties of Eisenstein Series

If \( k > 2 \), then \( G_k \in C^+(\Gamma(1), k) \), that is \( G_k \) is an entire modular form of weight \( k \) with multiplier system \( v \equiv 1 \). If \( k \) is odd then \( G_k(\tau) \equiv 0 \). This is obvious from the fact that if \( k \) is odd, then \( (c\tau + d)^{-k} + ((-c)\tau + (-d))^{-k} = 0 \). If \( k \) is even, then \( G_k(\tau) \) has the expansion

\[
G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)e^{2\pi in\tau}
\]

for \( \tau \in \mathcal{H} \), where \( \sigma_k = \sum_{d \mid n, d > 0} d^k \), and \( \zeta(k) \) is the Riemann zeta function, \( \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \).

The coefficients for this expansion are found using the cotangent expansion formula. [2] An outline of this calculation is given in Appendix D.

**Theorem 3.1.** If \( k \geq 4 \) then the normalized Eisenstein Series \( E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) \), where \( \zeta(k) \) is the Riemann zeta function.

**Proof.** This proof takes advantage of the fact that the lattice \( A = \{c\tau + d : c, d \in \mathbb{Z}, \text{not both zero}\} \), can be rewritten as \( A = \bigcup_{g=1}^{\infty} \{g(c'\tau + d') : c', d' \in \mathbb{Z}, (c', d') = 1\} \), and there are no points repeated. Therefore

\[
\sum_{c} \sum_{d} (c\tau + d)^{-k} = \sum_{g=1}^{\infty} \sum_{c} \sum_{d} (c\tau + d)^{-k} \quad \text{for} \quad (c, d) = g.
\]

Now consider \( G_k(\tau) = \sum_{c} \sum_{d} (c\tau + d)^{-k} \). Let the greatest common divisor of \( c \) and \( d \), \( (c, d) = g \). So then \((c\tau + d)^{-k} = \left(\frac{c}{g}\tau + \frac{d}{g}\right)^{-k} \cdot \frac{1}{g^k} \). Therefore

\[
G_k(\tau) = \sum_{c} \sum_{d} \left(\frac{c}{g}\tau + \frac{d}{g}\right)^{-k} \cdot \frac{1}{g^k} = \sum_{g=1}^{\infty} \sum_{c} \sum_{d} \left(\frac{c}{g}\tau + \frac{d}{g}\right)^{-k} \cdot \frac{1}{g^k} = \sum_{g=1}^{\infty} \frac{1}{g^k} \sum_{c'} \sum_{d'} (c'\tau + d')^{-k} = \sum_{c'} \sum_{d'} (c'\tau + d')^{-k} = 2\zeta(k) \left(2 \sum_{c'} \sum_{d'} (c'\tau + d')^{-k}\right) = 2\zeta(k)E_k(\tau)
\]

and \( E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) \). \( \square \)

This result gives us the expansion for the normalized Eisenstein series for \( k \) even and \( \tau \in \mathcal{H} \)

\[
E_k(\tau) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)e^{2\pi in\tau}.
\]

[5]
4 Discriminant Function

In this section we will discuss the discriminant function, \( \Delta(z) = e^{2\pi iz} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z})^{24} \) where \( z \in \mathcal{H} \). Before we discuss the properties of \( \Delta(z) \) it is necessary to briefly define some properties of infinite products.

4.1 Infinite Products [3] [5]

The infinite product \( \prod_{v=1}^{\infty} u_v = u_1 u_2 \cdots \) is defined to converge if for \( v > m \), no \( u_v = 0 \) and the limit \( U_m = \lim_{n \to \infty} (u_{m+1} u_{m+2} \cdots u_n) \) exists and has a finite value not equal to zero. The number \( U = u_1 u_2 \cdots u_m U_m \) is taken as the value of the infinite product. This product converges if and only if for any \( \epsilon > 0 \) there exists an \( n_0 \) such that for all \( n > n_0 \), \( |u_{n+1} u_{n+2} \cdots u_{n+r} - 1| < \epsilon \). In order to clarify this property we often write infinite products as \( \prod_{v=1}^{\infty} (1 + c_v) \) where \( u_v = 1 + c_v \). Then the property emerges that for \( c_v \geq 0 \), the infinite product \( \prod_{v=1}^{\infty} (1 + c_v) \) converges if and only if the sum \( \sum_{v=1}^{\infty} c_v \) converges. Furthermore, if \( \{f_v(z)\} \) is an infinite sequence of functions which are regular in some region \( R \). Then if \( \sum_{v=1}^{\infty} |f_v(z)| \) is uniformly convergent in every closed subregion \( R' \) of \( R \), then the infinite product is convergent in the entire region \( R \) and is a regular function \( f(z) \) in \( R \).

4.2 Properties of \( \Delta(z) \)

Based on what was discussed about infinite products the following is immediately obvious about \( \Delta(z) \). First it is clear that \( \Delta(z) \) is holomorphic in \( \mathcal{H} \), since the sum \( \sum_{m=1}^{\infty} | - e^{2\pi i m z} | \) is uniformly convergent in all closed subregions of \( \mathcal{H} \). Second, \( \Delta(z) \neq 0 \) in \( \mathcal{H} \), since \( e^{2\pi i m z} \neq 1 \) unless \( \text{Im}(z) = 0 \). Finally note that \( \Delta(z + 1) = \Delta(z) \) since \( e^{2\pi i m(z+1)} = e^{2\pi i m z} e^{2\pi i m} = e^{2\pi i m z} \).

\( \Delta(z) \) is a cusp modular form of weight 12 and multiplier system \( v \equiv 1 \). The Fourier expansion of \( \Delta(z) \) at \( \infty \) is \( \Delta(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \) for \( z \in \mathcal{H} \). The coefficients are referred to as the Ramanujan tau function, \( \tau(n) \), as Ramanujan conjectured the following about \( \tau(n) \):

\[
|\tau(p)| \leq 2 \cdot p^{11/2},
\]

\[
\tau(mn) = \tau(m) \tau(n) \text{ if } \gcd(m, n) = 1, \text{ and}
\]

\[
\tau(p^n) = \tau(p) \tau(p^{n-1}) + p^{1+ \tau(p-1)} \text{ if } p \text{ is prime and } n \geq 1.
\]

Finally it is interesting to note that \( 12^3 \Delta(z) = E_4(z)^3 - E_6(z)^2 \). It is easy to see that \( E_4(z)^3 - E_6(z)^2 \) is a cusp modular form of weight twelve since the \( n = 0 \) terms of the series (the ones pulled out of the sum) will cancel when subtracted. We will prove later that \( \dim \mathcal{C}^0(\Gamma(1), 12) = 1 \) and we know that \( \Delta(z) \) is a cusp form of weight twelve so it must be true that \( E_4(z)^3 - E_6(z)^2 \) is a multiple of \( \Delta(z) \).
5 Applications to Vector Spaces

What we know about the Eisenstein series and the discriminant function allows us to say several things about the dimensions of the vector spaces of all cusp modular forms and entire modular forms of a specific weight $k$. The goal of this section is to create a formula that will allow us to know the dimension of $C^0(\Gamma(1), k)$ and $C^+(\Gamma(1), k)$ for all even $k$.

First for $k \geq 4$, $k$ even, we know that $\dim C^+(\Gamma(1), k) \geq 1$. Since we know the Eisenstein series of weight $k$ is an entire modular form of weight $k$, so there is at least one function in this vector space. Knowing the dimension of the entire form vector space allows us to determine the dimension of the cusp form vector space. This is evident from the fact that any entire modular form can be written as a linear combination of a cusp form and the Eisenstein series of the appropriate weight as shown below.

**Theorem 5.1.** For $k \geq 4 \dim C^+(\Gamma(1), k) = 1 + \dim C^0(\Gamma(1), k)$.

**Proof.** [5] Consider 

$$f(\tau) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi in\tau} \quad \text{and} \quad G_k(\tau) = 2\zeta(k) + \sum_{n=1}^{\infty} b_n e^{2\pi in\tau}$$

where $f(\tau) - \frac{a_0}{2\zeta(k)} G_k(\tau) \in C^0(\Gamma(1), k)$. Now suppose that $C^0(\Gamma(1), k)$ has dimension $j$ and basis $B = \{f_1, f_2, ..., f_j\}$. We can then write $f(\tau) - \frac{a_0}{2\zeta(k)} G_k(\tau)$ as a linear combination of functions in this basis, $f(\tau) - \frac{a_0}{2\zeta(k)} G_k(\tau) = c_1 f_1 + c_2 f_2 + \cdots + c_j f_j$. So then $f(\tau) = \frac{a_0}{2\zeta(k)} G_k(\tau) + c_1 f_1 + c_2 f_2 + \cdots + c_j f_j$. So then since $f(\tau) \in C^+(\Gamma(1), k)$ we can write any entire modular form of weight $k$ as a linear combination of these $j + 1$ functions. Therefore $\{G_k, f_1, f_2, ..., f_j\}$ is a basis for $C^+(\Gamma(1), k)$ and $\dim C^+(\Gamma(1), k) = 1 + \dim C^0(\Gamma(1), k)$. $\square$

Next we will prove that there are no cusp forms of with an even weight less than twelve.

**Theorem 5.2.** For $k = 2, 4, 6, 8, 10 \dim C^0(\Gamma(1), k) = 0$.

**Proof.** [5] Suppose $f(\tau) \in C^0(\Gamma(1), k)$ for $k = 2$ without loss of generality. Now consider the function $\frac{f(\tau)}{\Delta(\tau)}$. Note that this function is holomorphic in the upper half plane since $\Delta(\tau) \neq 0$ in the upper half plane. Next note that when expanded at $\infty$

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi in\tau}$$

and

$$\Delta(\tau) = \sum_{n=1}^{\infty} c(n) e^{2\pi in\tau}$$

$$= e^{2\pi i\tau} + \sum_{n=2}^{\infty} c(n) e^{2\pi in\tau}$$

$$= e^{2\pi i\tau} \left( 1 + \sum_{n=1}^{\infty} c(n) e^{2\pi in\tau} \right)$$
The fact that \( c(1) = 1 \) follows because the only term that will have \( e^{2\pi i \tau} \) is the term that comes from all the first terms in the binomial being multiplied together when the product

\[
\Delta(\tau) = e^{2\pi i z} \prod_{m=1}^{\infty} (1 - e^{2\pi im\tau})^{24}
\]

is multiplied out. Therefore,

\[
\frac{f(\tau)}{\Delta(\tau)} = \frac{a_1 e^{2\pi i \tau} + a_2 e^{4\pi i \tau} + a_3 e^{6\pi i \tau} + \cdots}{e^{2\pi i \tau} (1 + c(1)e^{2\pi i \tau} + c(2)e^{4\pi i \tau} + \cdots)}
\]

\[
= \frac{a_1 + a_2 e^{2\pi i \tau} + a_3 e^{4\pi i \tau} + \cdots}{1 + c(1)e^{2\pi i \tau} + c(2)e^{4\pi i \tau} + \cdots}
\]

\[
= a_1 + \sum_{n=1}^{\infty} \alpha_n e^{2\pi in \tau}.
\]

Since \( \frac{f(\tau)}{\Delta(\tau)} \) is a modular form of weight two divided by a modular form of weight twelve, by Lemma 2.1, it has weight \( 2 - 12 = -10 \). Additionally since this modular form starts with the \( n = 0 \) term, it is an entire form. However by Theorem 2.1 there are no entire modular forms of weight less than zero. So \( \frac{f(\tau)}{\Delta(\tau)} \equiv 0 \), and since this function is holomorphic we know that \( f(\tau) \equiv 0 \). So there are no cups forms of weight two and \( \dim C^0(\Gamma(1), 2) = 0 \). So therefore for \( k = 2, 4, 6, 8, 10 \) \( \dim C^0(\Gamma(1), k) = 0 \).

By the same logic as the proof to Theorem 5.2 we can say that any cusp form of weight twelve divided by \( \Delta(\tau) \) is an entire form of weight zero, and a constant. Therefore all cusp forms of weight twelve are multiples of \( \Delta \) and so \( \dim C^0(\Gamma(1), 12) = 1 \).

**Theorem 5.3.** \( \dim C^+ (\Gamma(1), 2) = 0 \).

**Proof.** [5] We will first prove a transformation law for the anti-derivatives of modular forms of weight two. Let \( f(\tau) \in C^+ (\Gamma(1), 2) \). So we can write the Fourier expansion of \( f \) at infinity as \( f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi in \tau} \). Then define the anti-derivative of \( f \) to be

\[
F(\tau) = a_0 \tau + \sum_{n=1}^{\infty} \frac{a_n}{2\pi in} e^{2\pi in \tau}.
\]

By the chain rule, we also know that

\[
\frac{d}{d\tau} F(M\tau) = f(M\tau) \frac{d}{d\tau} (M\tau)
\]

\[
= f(M\tau) \frac{d}{d\tau} \left( \frac{a\tau + b}{c\tau + d} \right)
\]

\[
= f(M\tau) \left( \frac{a(c\tau + d) - c(a\tau + b)}{(c\tau + d)^2} \right)
\]

\[
= f(M\tau) \left( \frac{ad - bc}{(c\tau + d)^2} \right)
\]

\[
= f(M\tau) \left( \frac{1}{(c\tau + d)^2} \right)
\]
since \( ad - bc = 1 \) for all modular forms. Then since this modular form has multiplier system \( v \equiv 1 \), by the transformation law of modular forms \( f(M\tau) = (c\tau + d)^2 f(\tau) \). so then \( \frac{d}{d\tau} F(M\tau) = f(\tau) \) and after integrating \( F(M\tau) = F(\tau) + p_M \) where \( p_M \in C \) is a constant that depends the matrix.

We can write the Fourier expansion of \( f \) at infinity as \( f(\tau) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau} \). We know that \( f \) has anti-derivative \( F(\tau) = a_0 \tau + \sum_{n=1}^{\infty} \frac{a_n}{2\pi in} e^{2\pi in \tau} \), and based on the transformation law we determined above we know that \( F(M\tau) = F(\tau) + p_M \) for all \( M \in \Gamma(1) \). We want to prove that \( p_M = 0 \) for all \( M \in \Gamma(1) \).

Consider \( M = T \) as defined in Section 1.2.1 which gives \( M\tau = -\frac{1}{\tau} \). So \( F(T\tau) = F(\tau) + p_T \) and also \( F(T(T\tau)) = F(\tau) = F(T\tau) + p_T \). So \( F(\tau) = F(T\tau) + p_T \) and \( p_T = 0 \).

Now consider \( M = ST \) where \( S \) is the matrix defined in Section 1.2.1. By the transformation law we know \( F(ST\tau) = F(\tau) + p_{ST} \), that \( F(ST(ST\tau)) = F(ST\tau) + p_{ST} \), and that \( F(ST((ST)^2\tau)) = F((ST)^2\tau) + p_{ST} \). Recall that \((ST)^3 = I\) so then we know that \( F(\tau) = F((ST)^2\tau) + p_{ST} \). We can then substitute to get \( F(\tau) = (F((ST)\tau) + p_{ST}) + p_{ST} = (F(ST) + p_{ST}) + 2p_{ST} = F(\tau) + 3p_{ST} \). So then \( p_{ST} = 0 \).

So we have \( F(T\tau) = F(\tau) \) and \( F(ST\tau) = F(\tau) \). Then \( F(STT\tau) = F(T\tau) = F(\tau) \). But \( STT = S \) since \( T^2 = I \), so therefore \( F(S\tau) = F(\tau + 1) = F(\tau) \). Now consider \( F(\tau + 1) \)

\[
F(\tau + 1) = a_0(\tau + 1) + \sum_{n=1}^{\infty} \frac{a_n}{2\pi in} e^{2\pi in(\tau + 1)} = a_0 \tau + a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2\pi in} e^{2\pi in \tau} = a_0 \tau + F(\tau).
\]

So \( F(\tau) = a_0 \tau + F(\tau) \) and \( a_0 = 0 \). But then the first nonzero term of the Fourier series expansion has \( n > 0 \) so \( f \) is a cusp form. But by Theorem 5.2 we know that there are no cusp forms of weight two. So therefore there are no entire forms of weight two and \( \dim C^+(\Gamma(1), 2) = 0 \).

Using this fact, we can use the same logic as used in Theorem 5.2 to say that any cusp form \( f \) of weight fourteen divided by \( \Delta \) is an entire form of weight two. Therefore \( \frac{f(\tau)}{\Delta(\tau)} \equiv 0 \), and \( f(\tau) \equiv 0 \). Therefore there are no cusp forms of weight fourteen and \( \dim C^0(\Gamma(1), 14) = 0 \).

### 5.1 A General Formula

By combining all of these facts we can establish a general formula for \( \dim C^0(\Gamma(1), k) \) and \( \dim C^+(\Gamma(1), k) \).

**Theorem 5.4.** Let \( k \) be and even integer. Then

(i) For \( k \geq 4 \) we know

\[
\dim C^0(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k}{12} \right\rfloor & k \not\equiv 2 \pmod{12}.
\end{cases}
\]

(ii) For \( k \geq 4 \) we know

\[
\dim C^+(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k}{12} \right\rfloor & k \not\equiv 2 \pmod{12}.
\end{cases}
\]
(ii) For \( k \geq 0 \) we know
\[
\dim C^+(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \pmod{12} 
\end{cases}
\]

Proof. [5] We know the dimensions of the vector spaces of all cusp forms of weight 4, 6, 8, 10, 12, and 14. So then by Theorem 5.1 we know the dimensions of the vector spaces of entire forms of weights 4, 6, 8, 10, 12 and 14. Therefore we know both parts are true for \( k \leq 14 \). Now suppose \( k \geq 14 \) and that part (ii) holds for all weights less than \( k \). If \( f(\tau) \in C^0(\Gamma(1), k) \) then using the same proof technique as in Theorem 5.2 we know that \( \frac{f(\tau)}{\Delta(\tau)} \in C^+(\Gamma(1), k - 12) \). So therefore \( \dim C^0(\Gamma(1), k) = \dim C^+(\Gamma(1), k - 12) \). \( k - 12 < k \) so we know that part (ii) holds and so
\[
\dim C^0(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k-12}{12} \right\rfloor & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k-12}{12} \right\rfloor + 1 & k \not\equiv 2 \pmod{12} 
\end{cases}
\]
But \( \frac{k-12}{12} = \frac{k}{12} - 1 \) so therefore,
\[
\dim C^0(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k}{12} \right\rfloor & k \not\equiv 2 \pmod{12} 
\end{cases}
\]
Then by Theorem 5.1 and our knowledge about \( \dim C^+(\Gamma(1), k) \) for \( k = 0, 2 \) we can say that For \( k \geq 0 \) we know
\[
\dim C^+(\Gamma(1), k) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \pmod{12} \\
\left\lfloor \frac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \pmod{12} 
\end{cases}
\]

\[ \square \]

References


Appendix A  \( T^2 = (ST)^3 = I \)

\[
T^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{I} \\
= I.
\]

\[
(ST)^3 = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^3 \\
= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^3 \\
= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \\
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{I} \\
= I.
\]

Appendix B  Theorem 2.1

If \( F(\tau) \) is an entire or cusp modular form with weight less than zero then \( F(\tau) \equiv 0 \). The proof given below is for modular forms on the full modular group.

This proof makes use of the fact that \(|F(\tau)| = |F(x+iy)| \leq Jy^{-k} \) for all \( \tau = x+iy \in \mathcal{H} \), where \( J > 0 \) is independent of \( \tau \). In order to prove this fact we will consider the function

\[
\phi(\tau) = \phi(x+i y) \\
= |y^{\frac{k}{2}} F(x + iy)|.
\]
Next note that for any $M \in \Gamma$ where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$\Im(M\tau) = \Im\left(\frac{\alpha(x + iy) + \beta}{\gamma(x + iy) + \delta}\right)$$

$$= \Im\left(\frac{(\alpha(x + iy) + \beta)(\gamma(x - iy) + \delta)}{(\gamma(x + iy) + \delta)(\gamma(x - iy) + \delta)}\right)$$

$$= \frac{\alpha\delta y - \beta\gamma y}{|\gamma\tau + \delta|^2}$$

$$= \frac{y}{|\gamma\tau + \delta|^2}.$$

Now consider $\phi(M\tau)$ and let $F(\tau)$ be an entire modular form of weight $k$ on $\Gamma(1)$. Recall from Section 1.4 that $|v(M)| = 1$ for all $M \in \Gamma$.

$$\phi(M\tau) = \left|\left(\frac{y}{|c\tau + d|^2}\right)^{\frac{k}{2}} F(M\tau)\right|$$

$$= \left|\left(\frac{y}{|c\tau + d|^2}\right)^{\frac{k}{2}} v(M)(c\tau + d)^k F(\tau)\right|$$

$$= |y^{\frac{k}{2}} F(\tau)|$$

$$= \phi(\tau).$$

Now consider the expansion of $F$ at infinity, $F(\tau) = \sum_{n+\kappa \geq 0} a_n e^{2\pi i (n+\kappa)\tau/\lambda}$. Recall that in $\Gamma(1)$ the width of infinity, $\lambda = 1$. So then for $\tau \in \mathcal{H}$.

$$\phi(\tau) = |y^{\frac{k}{2}} F(\tau)|$$

$$= y^{\frac{k}{2}} \left|\sum_{n+\kappa \geq 0} a_n e^{2\pi i (n+\kappa)\tau}\right|$$

$$= y^{\frac{k}{2}} e^{-2\pi i (n_0+\kappa)y} \left|\sum_{n_0-n \geq 0} a_n e^{2\pi i (n_0-n)\tau}\right|$$

where $n_0$ is the smallest $n$ such that $a_n \neq 0$. As $y$ approaches infinity the sum converges to $a_n$, since $\lim_{y \to \infty} e^{2\pi i (n-n_0)\tau} = \lim_{y \to \infty} e^{2\pi i (n-n_0)\tau} e^{-2\pi i (n-n_0)y} = 0$ unless $n = n_0$. Additionally $\lim_{y \to \infty} y^{\frac{k}{2}} e^{-2\pi i (n_0+\kappa)y} = 0$ since if $n_0 + \kappa \neq 0$ the exponential shrinks very quickly and if $n_0 + \kappa = 0$ then the term $y^{\frac{k}{2}}$ goes to zero when $k < 0$. Therefore as $\tau \to \infty$, $\phi(\tau) \to 0$. Now consider the region $R^* = \overline{R} \setminus D_{\infty}$ where $\overline{R}$ is the closure of the standard fundamental region of $\Gamma(1)$ and $D_{\infty}$ is an open neighborhood of $\infty$. Then since $\phi$ is continuous and $R^*$ is compact, $\phi(\tau)$ must be bounded in $R^*$. Therefore there exists some $C$ such that $|\phi(\tau)| \leq C$ for all $\tau \in \mathcal{H}$. So then $|F(\tau)| \leq y^{-\frac{k}{2}} C$. We will now consider the coefficients of $F(\tau)$ Fix
\[ \tau = x + iy \in \mathcal{H} \text{ and let } \lambda \in \mathbb{R} \text{ Then} \]
\[
\int_{\tau}^{\tau + \lambda} F(\zeta)e^{-2\pi i(n+\kappa)\zeta}d\zeta = \int_{\tau}^{\tau + \lambda} \left( \sum_{m+\kappa \geq 0} a_m e^{2\pi i(m+\kappa)\zeta} \right) e^{-2\pi i(n+\kappa)\zeta}d\zeta
\]
\[
= \sum_{m+\kappa \geq 0} a_m \int_{\tau}^{\tau + \lambda} e^{2\pi i(m-n)\zeta}d\zeta
\]
\[
= \lambda a_n
\]
since \[ \int_{\tau}^{\tau + \lambda} e^{2\pi i(m-n)\zeta}d\zeta = 0 \text{ if } m \neq n \text{ and is equal to } \lambda \text{ when } m = n. \]
Therefore \[ a_n = \frac{1}{\lambda} \int_{\tau}^{\tau + \lambda} F(\zeta)e^{-2\pi i(n+\kappa)\zeta}d\zeta \]
and so by the ML inequality \[ |a_n| \leq C y^{-k} e^{2\pi i(n+\kappa)y} \]
and when \( k \) is negative \( a_n = 0 \) as \( y \to 0. \)

**Appendix C**  
**Lemma 2.1**  
Consider \( F_1 \in \{ \Gamma, k_1, v_1 \} \) and \( F_2 \in \{ \Gamma, k_2, v_2 \} \). Then
\[
F_1 F_2(M\tau) = F_1(M\tau) F_2(M\tau)
\]
\[
= (v_1(M)(c, d) \Gamma_1(\tau)) (v_2(M)(c, d) \Gamma_2(\tau))
\]
\[
= v_1(M) v_2(M)(c, d) \Gamma_1 + \kappa \Gamma_2 F_1(\tau) F_2(\tau)
\]
\[
= v_1 v_2(M)(c, d) \Gamma_1 + \kappa \Gamma_2 F_1 F_2(\tau).
\]

*what remains to complete the proof is to show that \( v_1v_2 \) is in fact a multiplier system.

**Appendix D**  
**Coefficients for the Fourier Expansion of \( G_k \) [5]**

This calculation relies on the fact that \[ \pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1}{z + m} + \frac{1}{z - m} = \sum_{m=\infty}^{\infty} \frac{1}{z + m}. \]

[2]
\[
\pi \cot(\pi z) = \pi \left( \frac{\cos(\pi z)}{\sin(\pi z)} \right)
\]
\[
= i\pi \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1}
\]
\[
= i\pi \left( 1 - \frac{2}{e^{2\pi iz} - 1} \right)
\]
\[
= i\pi - \frac{2\pi i}{e^{2\pi iz} - 1}
\]
\[
= i\pi - 2\pi i \sum_{n=1}^{\infty} e^{2\pi inz}. \quad \ast \text{Geometric series formula.}
So then $\sum_{m=-\infty}^{\infty} \frac{1}{z+m} = i\pi - 2\pi i \sum_{n=1}^{\infty} e^{2\pi inz}$, and when we take the derivative of both sides we get $\sum_{m=-\infty}^{\infty} \frac{1}{(z+m)^2} = (2\pi i)^2 \sum_{n=1}^{\infty} ne^{2\pi inz}$. If we call this the $k = 2$ case, induction on $k$ gives us $\sum_{m=-\infty}^{\infty} \frac{1}{(z+m)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi inz}$. Now note that when $k$ is even:

$$G_k(\tau) = \sum_{c} \sum_{d} (c\tau + d)^{-k}$$

$$= 2\zeta(k) + 2 \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} (c\tau + d)^{-k}.$$