Calculation of Positronium Energy Levels at Order \( m \alpha^7 \)
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Introduction
The main purpose of the research project is to calculate the positronium energy levels at the order \( m \alpha^7 \).

Positronium is a bound state of an electron and its antiparticle, positron. The state of these two particles can be described using wave functions denoted by \( \Psi \). In quantum physics we make predictions of how a certain particle behaves while interacting with the environment using the Schrödinger equation—one of the fundamental equations of physics. The most common form of the equation is given below

\[
\left( \frac{\hbar}{2m} \frac{\partial^2}{\partial r^2} + V \right) \Psi = i\hbar \frac{d}{dt} \Psi
\]

where \( V \) is some potential energy function (for instance, Coulomb potential).

It turns out that when the particle's interaction with the environment is described by Coulomb potential, the wave function is normalizable (and gives physically meaningful solutions) only for particular values of particle's energy \( E \). We say that a particle with a particular energy \( E \) is at a particular energy level.

Our goal in this research project is to calculate the characteristic energy levels of positronium at the order \( m \alpha^7 \). The values of the position and electron's energies are affected by multiple interactions, which can be described using Feynman diagrams. The diagrams can then be transformed into mathematical equations using the Feynman rules.

During the Hackman Research period I applied the Feynman rules for non-relativistic quantum electrodynamics (NRQED) to calculate the contributions of the \( \alpha \)\(^7\) Darwin correction and \( \alpha \)\(^7\) relativistic kinetic energy correction to the positronium ground state (1S) and the first excited state (2S, 2P) energy levels.

Green’s Function
We start with the momentum space inhomogeneous Schrödinger equation

\[
\left( E - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial p^2} \right) G_{\text{eq}}(p, f) = \int dV \left[ p \left( \frac{\partial}{\partial f} \right) G_{\text{eq}}(p, f) + 2pe \left( \frac{\partial}{\partial p} \right) G_{\text{eq}}(p, f) \right]
\]

We note that the free Green’s function can be expressed as

\[
S = (E - \frac{\hbar^2}{2m})^{-1}
\]

and we use this fact to write the full Green’s function in the form

\[
(S^{-1} - V)G_{\text{eq}} = 1
\]

The full Green’s function can then be expressed as a scattering series that represents all possible particle propagation amplitudes

\[
G_E = S + SV S + SV SV S + \ldots
\]

Hostler was the first to solve the inhomogeneous equation (by performing the Fourier transform)\(^[1]\) Later Schwinger obtained an equivalent solution using the \( \Omega \) symmetry.\(^[2]\) However, Adkins made the solution simpler by using the scattering series\(^[3]\) to find the complete \( \text{Schrödinger-Coulomb Green function} \)

\[
G_{\text{eq}}(p, f) = \left( -2m \right) \left( \frac{2e^2}{\alpha} \frac{\partial}{\partial f} - \frac{8m \alpha}{3} \frac{\partial^2}{\partial f^2} + \frac{32m \alpha^2}{7} \frac{\partial^3}{\partial f^3} + \frac{16m \alpha^3}{7} \frac{\partial^4}{\partial f^4} \right) f
\]

We use this solution to represent 0 potential, 1 potential, and manypotential (MP) propagations in the mathematical representations of the Feynman diagrams.

Integration
In order to obtain the numerical value of the correction, we must evaluate the integral. We first integrate over the energy components of the momentum 4-vectors. Since the integrand contains multiple singularities, we apply the residue theorem to integrate over \( p \) and \( q \) energy components.

\[
\int f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, a_k)
\]

We proceed by switching to the spherical coordinates and doing some of the simpler integrals over polar and azimuthal angles. In case of the relativistic KE 1M-MP integral, this leaves us with a 8 dimensional integral.

We complete the process by performing numerical integration. A Mathematica routine generates the appropriate wave function, as well as the remainder of the integrand. We use VEGAS program (written in Fortran) to obtain the final numerical result.

Current Stage
Currently, we have calculated Darwin and relativistic KE corrections at \( \alpha \)\(^7\). There are several other types of corrections that contribute to the positronium energy levels at order \( m \alpha^7 \)(for instance, spin-orbit correction). Calculating the spin-orbit correction will allow us to compare the obtained positronium result with the currently available result for hydrogen energy levels at order \( m \alpha^7 \). This will serve as a good checkpoint in the calculation process. However, many other terms will have to be found to finish the calculation of positronium energy levels at order \( m \alpha^7 \).

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References

Motivation
Electrodynamic interaction is the predominant form of interaction between the electron and positron in their bound state. Therefore, positronium is useful for comparing the results obtained by the currently accepted mathematical models of quantum electrodynamics with the experimental measurements of the energy levels. In this project we use non-relativistic quantum electrodynamics (NRQED).

Feynman Rules
The Feynman Rules allow us to express Feynman diagrams mathematically.

\[
\begin{align*}
\text{KE 1P-MP Equation} \\
\text{We use Feynman rules to translate the diagram representing the 1P-MP relativistic KE contribution at } \alpha(n) \text{ into a mathematical equation:}
\end{align*}
\]

\[
\int \frac{d^4p}{(2\pi)^4} G(\epsilon p_\mu) \left( \frac{1}{E + p_0 - m - \epsilon} \right) G(\epsilon p_\mu)
\]

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