The Quantum Time of Arrival

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Abstract

The arrival time problem exists in quantum mechanics due to the wavelike behavior of the subatomic particles. In this thesis, we formulate mechanisms to calculate the time of arrival. Our initial approach is to treat time as a parameter similar to the standard Copenhagen interpretation. We then apply our methods to a free particle wavefunction and compute its time of arrival. Observing that the results matched closely with the classical prediction, we extended the mechanism for a non-parametric time where time is a dynamical variable such as position. We found that the non-parametric treatment of time resulted in values that were closer to the classical arrival time. In doing so we have found that treating time as a non-parametric quantum variable is viable under our definition.
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Chapter 1

Introduction

Imagine a quantum particle in one dimension, moving from left to right towards some detector positioned on its path. One can picture such a detector measuring the rightmost tail of the particle which arrives some time earlier than its leftmost tail. This is because subatomic particles show wavelike behavior such that even a single photon has some spread in its distribution in space-time and cannot be taken as a point particle. The time taken by the particle to reach the detector therefore is not precise.

In the macroscopic world, a baseball, after being thrown by a pitcher, arrives towards a batsman at some time, given appropriate initial velocity of the ball. Let the detector be a bat or a catcher’s glove. If the pitcher throws the ball at speed 30m/s and his distance to the batsman is 15m, the time of arrival of the ball detected by the bat is 0.5 seconds as predicted by Newtonian mechanics. If the batsman fails to make a detection, the ball goes unaffected in its path where the catcher detects it again at some later time. This time of arrival of the ball with respect to a detector cannot be computed easily in the microscopic
world where there is no ‘ball’.

In quantum mechanics, we deal with expectation values of operators. An operator is something that acts on a wavefunction and gives a measurement. For example, position is an operator that, if applied to a wavefunction, gives us the average value of position. It, however, does not give the true position of the particle for it has none before the measurement [1]. Numerous difficulties arise when one thinks of time as an operator and calculate its expectation value.

Firstly, in traditional quantum mechanics, time is not treated as an observable represented by self-adjoint operators. According to the Copenhagen interpretation [1], time is an independent parameter such that wavefunctions evolve in time the same way position is a function of time in Newtonian mechanics. Pauli [2] has expressed the difficulty of treating time as a self-adjoint operator canonically conjugate with the Hamiltonian with its incompatibility with the Hamiltonian’s semi-bounded spectrum. To overcome this argument, Muga [3] has noted two kinds of solutions. His first solution is to construct the time operator outside the Hilbert space, the mathematical domain of quantum mechanics. The second one is to construct self-adjoint time operators from the canonical commutation relationship with the Hamiltonian. We have followed the second route.

A motivation that time can be treated equally like space arises from relativistic mechanics. For example in relativistic quantum field theory, time and space are treated similar. However, it is done at the expense of reducing the position as a parameter itself [4]. The expectation value of the time operator demands symmetric treatment of time with position. Over hundred years of Quantum Mechanics, the problem of asymmetry between position
and time has resulted in the time of arrival problem often referred by some as “a dark spot in the perfection of quantum scheme.” [5]

In this paper we observe the works in extended Hamiltonian mechanics that successfully establish the commutation relationship between the time operator and the Hamiltonian. Motivated by this, we extend the traditional formalism to obtain the expectation value of the time by treating it symmetrically with position. We then acknowledge the effects of measurements on a wavefunction and devise mechanisms to calculate the time of arrival. Our initial approach to the time of arrival problem begins with two assumptions. We begin with assuming that the time variable is directed forward and is non-dynamic and hence, a parameter similar to the Copenhagen interpretation of time. After formulating mechanisms to calculate the time of arrival for the parametric time variable, we generalize our approach to the non-parametric case. We then compare the time of arrival calculated using the general approach to our previous methods. We observe that the general case where time is not treated as a mere parameter results in values that are closer to the classical prediction than the parametrically treated case. Finally, we attempt to provide a Copenhagen rationale for the non-parametric time and discuss the consequences of such treatment.
Chapter 2

Formalism

We begin with the formalism that treats space and time on equal footing. In Quantum mechanics, wavefunctions and operators are the two mathematical constructs that specify a system. The state of a system is specified by its wavefunction and an observable is represented by an operator. Mathematically, wavefunctions are some abstract vectors in the Hilbert space and operators are the linear transformations that act on them [1]. In order to treat time and space equally, the transformations caused by the time operator on any wavefunction has to be similar to the transformations due to the position operator.

In classical mechanics, a system is fully specified by a set of $2D$ independent position variables $x_i = x_1, x_2, ..., x_D, x'_1, x'_2, ..., x'_D$ [6]. The momentum $p_i$ of each variable can be calculated from the Lagrangian $L(x, x', t)$ such that

$$p_i = \frac{\partial L(x, x', t)}{\partial x'_i} = p_i(x, x', t),$$
where \(x'\) is the derivative of \(x\) with respect to time. In a general parametric method [6], all position variables plus time are parametrized by an unspecified parameter \(\beta\) in the following way,

\[
x_\mu = x_\mu(\beta), \quad p_\mu = p_\mu(\beta),
\]

(2.1)

where \(0 \leq \mu \leq 3\). The use of such extra parameter can be observed in special relativity where \(\beta\) is defined as the proper time. We let \(x_0 = ct\) to insure proper dimensionality. For notational ease, we assume \(c = 1\). The derivative with respect to \(\beta\) is given by

\[
\dot{x}_\mu = \frac{dx_\mu}{d\beta}, \quad \text{while,} \quad x'_\mu = \frac{dx_\mu}{dt}.
\]

Our motivation to treat time as a dynamic variable rather than a parameter stems from the successful application of this generalization in both Lagrangian and the Hamiltonian mechanics [7]. One such result of the extended Hamiltonian mechanics [7] is

\[
x'_\nu = \frac{\partial \mathcal{H}(x_\mu, p_\nu)}{\partial p_\nu}, \quad p'_\nu = -\frac{\partial \mathcal{H}(x_\mu, p_\nu)}{\partial x_\nu},
\]

for \(0 \leq \nu, \mu \leq 3\). Hence, \(p_0\) is the extended Hamiltonian \(\mathcal{H}(x, p)\). The extended Hamiltonian generalizes the Poisson brackets to include the time variable [7]. The extended Poisson brackets are given by

\[
\{x_\mu, x_\nu\} = 0, \quad \{x_\mu, p_\nu\} = \delta_{\mu\nu}, \quad \{p_\mu, p_\nu\} = 0,
\]
for $0 \leq \nu, \mu \leq 3$. The canonical quantization method [8] allows us to promote Poisson brackets to quantum commutators in the following way,

$$\{A, B\} \rightarrow \frac{[\hat{A}, \hat{B}]}{i\hbar}.$$ 

The extended Poisson brackets can thus be mapped to the following commutation relationships,

$$[\hat{x}_\mu, \hat{x}_\nu] = 0, \quad [\hat{x}_\mu, \hat{p}_\nu] = i\hbar \delta_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0. \quad (2.2)$$

where $\hat{\rho}_0$ is the Hamiltonian operator.

In this paper we will follow the Schrödinger picture of wave mechanics. We will deal with normalized wavefunctions in one spatial and time dimension that are solutions to the Schrödinger equation. The commutation relation $[\hat{x}, \hat{t}] = 0$ allows us to simultaneously make a measurement of the system of both observables in any order without affecting the expectation value of either observable [1]. For a vector $|\psi\rangle$

$$\langle t, x| \hat{t} |\psi\rangle = t \langle t, x|\psi\rangle, \quad \langle t, x| \hat{x} |\psi\rangle = x \langle t, x|\psi\rangle.$$ 

The state $|t, x\rangle$ is different from the conventional approach in quantum mechanics. Here the vector represents an eigenstate of both $\hat{x}$ and $\hat{t}$, while in the conventional quantum mechanics, something analogous to $|t, x\rangle$ would be the vector $|x(t)\rangle$ which refers to the eigenstate of $|x\rangle$ at the time $t$. 

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2.1 The Time Operator

Let $\psi(x,t)$ be a wavefunction. The time of arrival has to be calculated probabilistically. The average value of a function $f$ is given by

$$\langle f(j) \rangle = \sum_{j=0}^{\infty} f(j) P(j),$$

(2.3)

where $P(j)$ is the probability of event $j$. Thus, the average value of the time operator is

$$\bar{t} = \frac{\sum_{j=0}^{\infty} t_j |\psi(x_d,t_j)|^2}{\sum_{j=0}^{\infty} |\psi(x_d,t_j)|^2} = \sum_{j=0}^{\infty} t_j \rho(t_j),$$

(2.4)

where the $\rho(t_j)$ is the probability density of finding the particle at time $t_j$ at position $x_d$. We have

$$\rho(t_j) = \frac{|\psi(x_d,t_j)|^2}{\sum_{j=0}^{\infty} |\psi(x_d,t_j)|^2}.$$  

(2.5)

For continuous variable $t$, Eq (2.4) becomes

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t |\psi(x_d,t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(x_d,t)|^2 dt}.$$  

(2.6)

For a normalized wavefunction the value of the denominator is unity.

Similarly, the expectation value of position in traditional quantum mechanics at some time $t_a$ is given by
2.1 The Time Operator

\[ \bar{x} = \frac{\int \int x |\psi(x,t)|^2 \, dx}{\int \int |\psi(x,t)|^2 \, dx} \]. \quad (2.7)

Our time of arrival formulation begins with observing a projection operator \( P = |x_a, t_b \rangle \langle x_a, t_b| \) for some \( a \) and \( b \). The sum of the projection operators is 1, if we sum over a complete set of states. Therefore,

\[ \int \int |x,t \rangle \langle x,t| \, dx \, dt = 1. \quad (2.8) \]

The inner product of a vector \( |\psi \rangle \) can be written as

\[ \langle \psi | \psi \rangle = \int \int \int \int |\psi(x,t)|^2 \, dx \, dt = \int \int \int |\psi(x,t)|^2 \, dx \, dt. \quad (2.9) \]

Born’s statistical interpretation \([1]\) of wavefunctions allows us to interpret the term \( |\psi(x,t)|^2 \, dx \, dt \) as the probability of measuring the particle in the small region of space-time, \( x, x+dx \) and \( t, t+dt \). The probability density \( \rho(x,t) \) is given by

\[ \rho(x,t) \, dx \, dt = \frac{|\psi(x,t)|^2 \, dx \, dt}{\int \int |\psi(x,t)|^2 \, dx \, dt}. \quad (2.10) \]

where the denominator is a constant of normalization. Now, the probability over all space-time is

\[ \int \int \rho(x,t) \, dx \, dt = \frac{\int \int |\psi(x,t)|^2 \, dx \, dt}{\int \int |\psi(x,t)|^2 \, dx \, dt} = 1. \quad (2.11) \]

Let us consider a region of space-time \( \mathcal{A} \) where \( x_a - \frac{\epsilon}{2} \leq x \leq x_a + \frac{\epsilon}{2} \) and \( t_0 \leq t \leq t_f \).
Let $\rho_{\mathcal{A}}$ be the probability density of the wavefunction in this region. Then,

$$\rho_{\mathcal{A}}(x,t)\,dx\,dt = \frac{|\psi(x,t)|^2\,dx\,dt}{\int_{t_0}^{t_f} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2\,dx\,dt}. \tag{2.12}$$

We have chosen the normalization constant so that the probability of finding the particle over this region is unity. Thus,

$$\int_{t_0}^{t_f} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} \rho_{\mathcal{A}}(x,t)\,dx\,dt = 1. \tag{2.13}$$

The expectation value of $t$ in this region of space-time $\mathcal{A}$ is given by

$$\bar{t} = \int_{t_0}^{t_f} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} t \rho_{\mathcal{A}}(x,t)\,dx\,dt = \frac{\int_{t_0}^{t_f} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} t |\psi(x,t)|^2\,dx\,dt}{\int_{t_0}^{t_f} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2\,dx\,dt}. \tag{2.14}$$

When $\varepsilon \ll 1$ we can write

$$\int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} f(x)\,dx = f(x_a)\varepsilon, \tag{2.15}$$

for some continuous function $f(x)$. The R.H.S. of Eq. (2.15) is the area of length $\varepsilon$ times height $f(x_a)$ that gets closer to the L.H.S as $\varepsilon$ gets smaller. The constant $\varepsilon$ cancels out and the $\bar{t}$ in Eq. (2.14) becomes

$$\bar{t} = \frac{\int_{t_0}^{t_f} t |\psi(x_a,t)|^2\,dt}{\int_{t_0}^{t_f} |\psi(x_a,t)|^2\,dt}. \tag{2.16}$$

In the case when our time span is stretched so that $t_0$ and $t_f$ tend to $-\infty$ and $\infty$ respectively,

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t |\psi(x_a,t)|^2\,dt}{\int_{-\infty}^{\infty} |\psi(x_a,t)|^2\,dt}, \tag{2.17}$$
which is equivalent to Eq. (2.6).

Among the efforts in the construction of a "time of arrival" operator such as given by Eq. (2.17), numerous problems have been realized. According to Allcock [9], such time of arrival operator lies outside the Hilbert Space, beyond which many conventional techniques of quantum mechanics are inapplicable. Meanwhile, in Grot’s paper, he argues that constructing such an operator within the Hilbert Space involves decomposing the particle’s Heisenberg states into the eigenstates of this operator [5], which is a procedure of immense complication. Our approach is direct and mathematically simple. We begin with the one-dimensional free particle used in Grot’s paper.

### 2.2 Free particle wavefunction

Throughout the paper, we will be using the same non-relativistic free particle used by Grot [5]. The wavefunction $\psi_0(x,t)$ begins at position $x_0$ and time $t_0$ such that

$$\psi_0(x,t) = \left( \frac{\delta^2}{2\pi} \right)^{1/4} \frac{\exp\left[ -k_0^2 \delta^2 \right]}{\sqrt{\delta^2 + \frac{i\hbar}{2m}}} \exp \left[ \frac{(2\delta^2 k_0 + i(x-x_0))^2}{4\delta^2 + \frac{2i\hbar}{m}} \right],$$

(2.18)

where $m$ is the mass of the particle, $\hbar$ is the reduced Planck’s constant, $\delta$ is the width of the wave packet at time $t_0$, and $k_0$ is the wave number. We calculated the expectation values to be

$$\langle p(t) \rangle = \hbar k_0, \quad \langle \Delta p(t) \rangle = \frac{\hbar}{2\delta}.$$

(2.19)
\[ \langle x(t) \rangle = x_0 + \hbar k_0 t / m, \]
\[ \langle \Delta x(t) \rangle = \delta \sqrt{1 + \frac{t^2 \hbar^2}{4\delta^4 m^2}}, \] (2.20)

consistent with Grot. The Ehrenfest theorem [1] allows us to imagine the \( \psi_0(x,t) \) moving to the right with speed \( \hbar k_0 / m \). For simplicity we let \( m, \hbar = 1 \) so that \( k_0 \) can act as the wavefunction’s classical velocity.

We argue that applying Eq. (2.6) to \( \psi_0(x,t) \) gives the ”presence time” of the particle and not the time of arrival. The presence time can be calculated by putting a detector on the particle’s path at time \( t_1 \) and checking whether the particle was detected. The process is repeated for a large number \( n \) to get the probability of the particle’s detection at time \( t_1 \).

We repeat the process for time \( t_2 > t_1 \) and find the probability of detecting the particle at time \( t_2 \) and so on. The presence time is the summed average of all the probability densities multiplied by the time \( t_i \) given by Eq. (2.4). The presence time does not take into account the perturbations due to continuous measurement of the wavefunction [7]. In order to calculate the expectation value of time using Eq. (2.4), one has to continuously undo the effects of measurement on \( \psi(x,t) \). In our approach, we take into account the effects of measurement. In order to understand that, one has to understand first, the mechanics of time evolution.

### 2.3 Propagators

Let \( |\psi, t_0\rangle \) denote the vector at time \( t_0 \). It is not guaranteed that the wavefunction will remain at the same state at time \( t' > t \). A time evolution operator evaluates how a wave-
function’s state changes with time. It is given by [10],

\[ \mathcal{U}(t,t') = e^{-i\hat{H}(t'-t)/\hbar}. \]  

(2.21)

Here \( \hat{H} \) is the Hamiltonian, which is time independent. It is important to note that \( \mathcal{U}(t,t) = 1 \).

Additionally, we have

\[
\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}(t',t) \mathcal{U}(t,t') \\
= e^{-i\hat{H}(t'-t)/\hbar} e^{-i\hat{H}(t'-t)/\hbar} \\
= e^{-i\hat{H}(t'-t+t'-t)/\hbar}.
\]

Hence,

\[ \mathcal{U}^\dagger \mathcal{U} = 1. \]  

(2.22)

\( \mathcal{U}(t,t') \) being unitary guarantees that the normalized wavefunctions remain normalized later in time [10]. This fact becomes important later as we take into account the changes in the wavefunction’s states caused by measurements. The state of the wavefunction at some time \( t' > t_0 \) is given by

\[ |\psi,t'\rangle = \mathcal{U}(t_0,t') |\psi,t_0\rangle. \]

We have
\[\psi(x', t') = \langle x' | \psi, t' \rangle \]
\[= \langle x' | e^{-i\hat{H}(t' - t_0)/\hbar} | \psi, t_0 \rangle \]
\[= \int_{-\infty}^{\infty} \langle x' | e^{-i\hat{H}(t' - t_0)/\hbar} | x_0 \rangle \langle x_0 | \psi, t_0 \rangle dx_0 \]
\[= \int_{-\infty}^{\infty} \langle x' | e^{-i\hat{H}(t' - t_0)/\hbar} | x_0 \rangle \psi(x_0, t_0) dx_0 \]
\[= \int_{-\infty}^{\infty} \langle x', t' | x_0, t_0 \rangle \psi(x_0, t_0) dx_0. \]

Hence,

\[\psi(x', t') = \int_{-\infty}^{\infty} \mathcal{K}(x', t', x_0, t_0) \psi(x_0, t_0) dx_0, \quad (2.23)\]

where \(\mathcal{K}(x', t', x, t_0) = \langle x', t' | x_0, t_0 \rangle\) is the propagator. In the case when \(t' = t_0\),

\[\psi(x', t_0) = \int_{-\infty}^{\infty} \mathcal{K}(x', t_0, x_0, t_0) \psi(x_0, t_0) dx_0 \]
\[= \int_{-\infty}^{\infty} \langle x', t_0 | x_0, t_0 \rangle \psi(x_0, t_0) dx_0 \]
\[= \int_{-\infty}^{\infty} \langle x' | \mathcal{K}(t_0, t_0) | x_0 \rangle \psi(x_0, t_0) dx_0 \]
\[= \int_{-\infty}^{\infty} \langle x' | x_0 \rangle \psi(x_0, t_0) dx_0 \]
\[= \int_{-\infty}^{\infty} \delta(x' - x_0) \psi(x_0, t_0) dx_0 \]
\[= \psi(x', t_0). \]

For a free particle \([10]\)

\[\mathcal{K}(x', t', x_0, t_0) = \sqrt{\frac{m}{2\pi\hbar(t' - t_0)}} \exp \left[ \frac{im(x' - x_0)^2}{2\hbar(t' - t_0)} \right]. \quad (2.24)\]
Chapter 3

The Time of Arrival

We are now ready to formulate the mechanism which will allow us to calculate the time of arrival. We begin with the two assumptions that the time variable $t$ is directional and non-dynamic. Directionality is a constraint for $t$ in a sense that we are restricted by the forward arrow of time to detect the states that occur in the past. The directionality constraint will be discussed again after we define the past and the future states of a wavefunction. For now, $t_{n+1} > t_n$ for all $n$ and the measurements taken at time $t_{n+1}$ occur later than at time $t_n$. We also assume that the time variable is non-dynamic which implies that the detection of the $n^{th}$ state may occur only at time $t_n$. In dynamic case, as we will discuss later, the wavefunction remains in the superposition of all its detected states at all time $t$ in the range $t_0$ to $t_f$ and therefore, any state has a probability of being detected at any time $t_0 \leq t < t_f$. Here, $t_0 \leq t < t_f$ is the window when our wavefunction is in the interaction with the detector.
3.1 Measurements affecting the Wavefunction

Measurements in quantum mechanics radically affect the wavefunction. If a wavefunction collapses after a first measurement giving a sharp peak within our space-time detection window, we call it detected. Otherwise the wavefunction collapses to a non-detected state and evolves in accordance with the Schrödinger equation. Hence, whenever we make a measurement, the wavefunction ‘chooses’ a definite state. Let $\psi(x, t)$ be our wavefunction. In Dirac notation the wavefunction is indicated by states $|\psi_n, t_n\rangle$ after the $n^{th}$ measurement. Let $t_0$ be the beginning time.

3.1.1 First Measurement

For a time $t'$, the sum of the projection variables $|x, t'\rangle\langle x, t'|$ over all space is unity, i.e.

$$\int_{-\infty}^{\infty} |x, t'\rangle\langle x, t'| \, dx = 1. \quad (3.1)$$

The wavefunction $|\psi_0, t\rangle$ can be written as

$$|\psi_0, t\rangle = \int_{-\infty}^{\infty} \langle x, t'|\psi_0, t\rangle |x, t'\rangle \, dx. \quad (3.2)$$

We define the detected state $|\phi_1, t\rangle$ such that, if the first measurement is at time $t_1$,

$$|\phi_1, t\rangle = \frac{1}{c_1} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \langle x|\psi_0, t_1\rangle |x, t_1 - t\rangle \, dx, \quad (3.3)$$
where \( x_1 - \frac{\delta_1}{2} \leq x \leq x_1 + \frac{\delta_1}{2} \) is the spatial region where the detection takes place. The term \( c_1 \) is the normalization constant. Before the measurement (\( t < t_1 \)) the detected and not detected states are in superposition. For time \( t_0 \leq t < t_1 \) we can write \( |\psi_0, t\rangle \) as

\[
|\psi_0, t\rangle = a_1 |\phi_1, t\rangle + (|\psi_0, t\rangle - a_1 |\phi_1, t\rangle),
\]

(3.4)

where \( |\psi_0, t\rangle - a_1 |\phi_1, t\rangle \) is the non-detected state and \( a_1 \) is the amplitude of detection. We observe that

\[
\langle \phi_1, t | \phi_1, t \rangle = \begin{cases} 
\frac{1}{|c_1|^2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \langle \psi_0, t_1 | x' \rangle \langle x', t_1 - t | x, t_1 - t \rangle \langle x | \psi_0, t_1 \rangle \, dx' \, dx \\
0 
\end{cases}
\]

\[
= \frac{1}{|c_1|^2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \psi_0^*(x', t_1) \psi_0(x, t_1) \langle x', t_1 - t | x, t_1 - t \rangle \, dx' \, dx \\
= \frac{1}{|c_1|^2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} \psi_0^*(x', t_1) \psi_0(x, t_1) \delta(x - x') \, dx' \, dx.
\]

There are two cases,

\[
\langle \phi_1, t | \phi_1, t \rangle = \begin{cases} 
\frac{1}{|c_1|^2} \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} |\psi_0(x, t_1)|^2 \, dx & \text{if } x_1 - \frac{\delta_1}{2} \leq x \leq x_1 + \frac{\delta_1}{2} \\
0 & \text{otherwise.}
\end{cases}
\]

Since we are confining our detector to a range \( x_1 - \frac{\delta_1}{2} \leq x \leq x_1 + \frac{\delta_1}{2} \) we expect the probability of detection outside to be zero. We are free to normalize the detector function in the
range \( x_1 - \frac{\delta_1}{2} \leq x \leq x_1 + \frac{\delta_1}{2} \). We take
\[
\langle \phi_1, t | \phi_1, t \rangle = 1. \quad (3.5)
\]

Therefore,
\[
|c_1|^2 = \int_{x_1 - \frac{\delta_1}{2}}^{x_1 + \frac{\delta_1}{2}} |\psi_0(x, t_1)|^2 \, dx. \quad (3.6)
\]

We can also write Eq. (3.4) as
\[
|\psi_0, t \rangle = a_1 |\phi_1, t \rangle + b_1 |\psi_1, t \rangle, \quad (3.7)
\]

where \( |\psi_1, t \rangle \) is the state when \( |\psi_0, t \rangle \) is not detected. Here, \( b_1 \) is the amplitude of not detecting the particle. The probability of finding the particle in the range \( x_1 \pm \delta_1/2 \) at time \( t_1 \) is given by
\[
P_1(t_1) = |a_1|^2 \quad (3.8)
\]

Detecting and non-detecting the particle are two mutually exclusive events. They are also mutually exhaustive. Therefore, their probabilities add up in the following manner,
\[
|a_1|^2 + |b_1|^2 = 1. \quad (3.9)
\]

Now, from Eq. (3.7) we have
\[
|\psi_1, t \rangle = \frac{1}{b_1} \left( |\psi_0, t \rangle - a_1 |\phi_1, t \rangle \right). \quad (3.10)
\]
3.1 Measurements affecting the Wavefunction

At the time of measurement, the wavefunction collapses to either the detected or the non-detected state but not both. Thus, $|\phi_1, t\rangle$ and $|\psi_1, t\rangle$ are orthogonal at time $t_1$ i.e.

$$\langle \phi_1, t_1 | \psi_1, t_1 \rangle = 0. \quad (3.11)$$

Using Eq. (3.10) we get

$$\langle \phi_1, t_1 | \left( \frac{1}{b_1} (|\psi_0, t_1\rangle - a_1 |\phi_1, t_1\rangle) \right) = 0. \quad (3.10)$$

Hence,

$$\frac{1}{b_1} \left( \langle \phi_1, t_1 | \psi_0, t_1 \rangle - a_1 \langle \phi_1, t_1 | \phi_1, t_1 \rangle, t_1 \rangle \right) = 0. \quad (3.10)$$

Now from Eq. (3.9) $b_1 = 0$ implies that $a_1 = 1$ and the state $|\psi_1, t\rangle$ do not exist, since $b_1$ is the amplitude if the state $|\psi_1, t\rangle$. For $b \neq 0$, we have

$$\langle \phi_1, t_1 | \psi_0, t_1 \rangle - a_1 \langle \phi_1, t_1 | \phi_1, t_1 \rangle = 0. \quad (3.9)$$

Applying Eq. (3.5) and solving gives

$$a_1 = \langle \phi_1, t_1 | \psi_0, t_1 \rangle. \quad (3.12)$$
Using Eq. (3.3) we get the value of $a_1$ to be

$$a_1 = \frac{1}{c_1} \int_{x_1-\delta/2}^{x_1+\delta/2} \langle \psi_0, t_1 | x \rangle \langle x, | \psi_0, t_1 \rangle \, dx = \frac{1}{c_1} \int_{x_1-\delta/2}^{x_1+\delta/2} |\psi_0(x, t_1)|^2 \, dx.$$  

Applying Eq. (3.6) we obtain

$$a_1 = |c_1|. \quad (3.13)$$

Finally, we make sure that $|\psi_1, t_1 \rangle$ is normalized. Using Eq. (3.10),

$$\langle \psi_1, t_1 | \psi_1, t_1 \rangle = \frac{1}{|b_1|^2} \left( \langle \psi_0, t_1 | -a_1^* \langle \phi_1, t_1 \rangle \rangle (|\psi_0, t_1 \rangle - a_1 |\phi_1, t_1 \rangle) \right)$$

$$= \frac{1}{|b_1|^2} \left( \langle \psi_0, t_1 | \psi_0, t_1 \rangle - a_1^* \langle \phi_1, t_1 | \psi_0, t_1 \rangle - a_1 \langle \psi_0, t_1 | \phi_1, t_1 \rangle + |a_1|^2 \langle \phi_1, t_1 | \phi_1, t_1 \rangle \right).$$

Using Eq. (3.12), Eq. (3.5) and the fact that $\psi_0(x, t)$ is normalized we get

$$\langle \psi_1, t_1 | \psi_1, t_1 \rangle = \frac{1}{|b_1|^2} \left( 1 - a_1^* a_1 - a_1 a_1^* + |a_1|^2 \right) = \frac{1}{|b_1|^2} \left( 1 - |a_1|^2 \right) = 1. \quad (3.14)$$

### 3.1.2 Second Measurement

If the particle is not detected at time $t_1$, we make a second measurement at time $t_2$. During the time $t_1 \leq t < t_2$,

$$|\psi_1, t \rangle = a_2 |\phi_2, t \rangle + \left( |\psi_1, t \rangle - a_2 |\phi_2, t \rangle \right) = a_2 |\phi_2, t \rangle + b_2 |\psi_2, t \rangle, \quad (3.15)$$

where the state $|\psi_2, t \rangle$ is given by.
3.1 Measurements affecting the Wavefunction

\[ |\psi_{2},t\rangle = \frac{1}{b_2} (|\psi_{1},t\rangle - a_2|\phi_{2},t\rangle). \]  \hspace{1cm} (3.16)

Analogous to Eq. (3.30)

\[ |a_2|^2 + |b_2|^2 = 1. \]  \hspace{1cm} (3.17)

Here we define \(|\phi_{2},t\rangle\) similar to the first measurement, i.e.,

\[ |\phi_{2},t\rangle = \frac{1}{c_2} \int_{x_1-\delta/2}^{x_1+\delta/2} \langle x|\psi_{1},t_2\rangle |x,t_2-t\rangle dx. \]  \hspace{1cm} (3.18)

Using Eq. (3.10) we get

\[
|\phi_{2},t\rangle = \frac{1}{c_2} \int_{x_1-\delta/2}^{x_1+\delta/2} \left[ \frac{1}{b_1} \left( |\psi_{0},t_2\rangle - a_1|\phi_{1},t_2\rangle \right) \right] |x,t_2-t\rangle dx
= \frac{1}{c_2b_1} \int_{x_1-\delta/2}^{x_1+\delta/2} \left( \langle x|\psi_{0},t_2\rangle - a_1 \langle x|\phi_{1},t_2\rangle \right) |x,t_2-t\rangle dx
= \frac{1}{c_2b_1} \int_{x_1-\delta/2}^{x_1+\delta/2} \left( \psi_{0}(x,t_2) - a_1 \phi_{1}(x,t_2) \right) |x,t_2-t\rangle dx.
\]

From Eq. (3.3) we obtain the relationship,

\[ \phi_{1}(x,t_2) = \langle x,0|\phi_{1},t_2\rangle = \frac{1}{c_1} \int_{x_1-\delta/2}^{x_1+\delta/2} \langle x'|\psi_{0},t_1\rangle \langle x,0|x',t_1-t_2\rangle dx'. \]

Hence,

\[ \phi_{1}(x,t_2) = \frac{1}{c_1} \int_{x_1-\delta/2}^{x_1+\delta/2} \psi_{0}(x',t_1).\phi'(x,t_2,x',t_1) dx'. \]  \hspace{1cm} (3.19)
3.1 Measurements affecting the Wavefunction

where \( \mathcal{H}(x,t_2,x',t_1) \) is the time evolution propagator studied in Sec. (2.3). Now, from Eq. (3.18)

\[
\langle \phi_2, t | \phi_2, t \rangle = \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \langle \psi_1, t_2 | x' \rangle \langle x | \psi_1, t_2 \rangle \langle x', t_2 - t | x, t_2 - t \rangle \, dx' \, dx
\]

\[
= \frac{1}{|c_2|^2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \psi_1^*(x',t_2) \psi_1(x,t_2) \langle x', t_2 - t | x, t_2 - t \rangle \, dx' \, dx
\]

\[
= \frac{1}{|c_2|^2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} \psi_1^*(x',t_2) \psi_1(x,t_2) \delta(x-x') \, dx' \, dx
\]

\[
= \frac{1}{|c_2|^2} \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} |\psi_1(x,t_2)|^2 \, dx,
\]

in the range \( x_1 \pm \delta_1/2 \). We normalize \( \phi_2(x,t) \) by requiring

\[
\langle \phi_2, t | \phi_2, t \rangle = 1. \tag{3.20}
\]

Therefore,

\[
|c_2|^2 = \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} |\psi_1(x,t_2)|^2 \, dx. \tag{3.21}
\]

As in Eq. (3.11), the two states, detected and not-detected, have to be orthogonal at time \( t_2 \). Thus,

\[
\langle \phi_2, t_2 | \psi_2, t_2 \rangle = 0. \tag{3.22}
\]
Expanding $|\psi_{2}, t_{2}\rangle$ using Eq. (3.16) gives

$$
\langle \phi_{2}, t_{2} | \left[ \frac{1}{b_{2}} \left( |\psi_{1}, t_{2}\rangle - a_{2}|\phi_{2}, t_{2}\rangle \right) \right] = 0.
$$

From Eq. (3.20) we solve for $a_{2}$ to get

$$
a_{2} = \langle \phi_{2}, t_{2} | \psi_{1}, t_{2}\rangle. \tag{3.23}
$$

We use the definition of $|\phi_{2}, t_{2}\rangle$ from Eq. (3.18) we get

$$
a_{2} = \frac{1}{c_{2}} \int_{x_{1}-\delta_{1}/2}^{x_{1}+\delta_{1}/2} \langle \psi_{1}, t_{2}|x\rangle \langle x|\psi_{1}, t_{2}\rangle \, dx
$$

$$
= \frac{1}{c_{2}} \int_{x_{1}-\delta_{1}/2}^{x_{1}+\delta_{1}/2} |\psi_{1}(x, t_{2})|^2 \, dx
$$

$$
= \frac{1}{c_{2}} |c_{2}|^2
$$

$$
= |c_{2}|.
$$

Finally, we check whether $|\psi_{2}, t_{2}\rangle$ is normalized. The inner product of $|\psi_{2}, t_{2}\rangle$ with itself is given by dotting the bra-kets of Eq. (3.16)

$$
\langle \psi_{2}, t_{2} | \psi_{2}, t_{2}\rangle = \frac{1}{|b_{2}|^2} \left( \langle \psi_{1}, t_{2}| - a_{2}^* \langle \phi_{2}, t_{2}| \right) \left( |\psi_{1}, t_{2}\rangle - a_{2}|\phi_{2}, t_{2}\rangle \right)
$$

$$
= \frac{1}{|b_{2}|^2} \left( \langle \psi_{1}, t_{2} | \psi_{1}, t_{2}\rangle - a_{2}^* \langle \phi_{2}, t_{2} | \psi_{1}, t_{2}\rangle - a_{2} \langle \psi_{1}, t_{2} | \phi_{2}, t_{2}\rangle + |a_{2}|^2 \langle \phi_{2}, t_{2} | \phi_{2}, t_{2}\rangle \right).$$
Applying Eq. (3.14) and Eq. (3.20) we get

\[ \langle \psi_2, t_2 | \psi_2, t_2 \rangle = \frac{1}{|b_2|^2} \left( 1 - a_2^* a_2 - a_2^* a_2 + |a_2|^2 \right). \]

Observing that \( a_2 = \langle \phi_2, t_2 | \psi_1, t_2 \rangle \), we have

\[ \langle \psi_2, t_2 | \psi_2, t_2 \rangle = \frac{1}{|b_2|^2} \left( 1 - a_2^* a_2 - a_2 a_2^* + |a_2|^2 \right) = \frac{1}{|b_2|^2} \left( 1 - |a_2|^2 \right) = 1. \]

The probability of finding the particle in the range \( x_1 \pm \delta_1 / 2 \) at time \( t_2 \) is given by

\[ P_2(t_2) = |a_2|^2. \quad (3.24) \]

However, since we do not detect the particle in the first measurement, the probability that we detect it on the second measurement is dependent on the first. Here, \( P_2(t_2) \) is a conditional probability of detecting the particle at time \( t_2 \) given we do not detect it at time \( t_1 \). The true probability \( \pi(t_2) \) is the probability of detecting the particle at time \( t_2 \) and not detecting it at time \( t_1 \) given by the Kolmogorov definition of conditional probabilities,

\[ P(A \cap B_{not}) = P(A/B_{not})P(B_{not}), \quad (3.25) \]
for two events A and B. Hence, if the events A and B denote detection at times \( t_2 \) and \( t_1 \),

\[
\pi(t_2) = P_2(t_2) \left( 1 - P_1(t_1) \right)
= a_2^2(t_2) b_1^2(t_1).
\]

We can perform similar procedures for time \( t_2 \leq t < t_3 \) and so on. In general, for \( t_{n-1} \leq t < t_n \),

\[
|\psi_{n-1}, t \rangle = a_n |\phi_n, t \rangle + b_n |\psi_n, t \rangle, \tag{3.26}
\]

\[
|a_n|^2 + |b_n|^2 = 1, \tag{3.27}
\]

\[
\pi(t_n) = \prod_{i=1}^{n-1} b_i^2(t_i) a_n^2(t_n). \tag{3.28}
\]

Finally, the time of arrival for the directional and non-dynamic time variable is deduced from similar arguments that led to Eq. (2.4)

\[
\bar{t} = \frac{\sum_{j=1}^{n} t_j \pi(t_j)}{\sum_{j=1}^{n} \pi(t_j)}, \tag{3.29}
\]

which is equivalent to Eq. (2.16).
3.1 Measurements affecting the Wavefunction

### 3.1.3 Solving for the amplitudes

Using the wavefunction in Eq. (2.18) and Eq. (3.6), we can solve for the value of $a_1$ to get

$$a_1 = \frac{1}{2} \left[ -\text{erf} \left( \frac{\sqrt{2} \delta \left( t_1 - t_0 \right) k_0 + x_0 - x_1 - \frac{\delta_1}{2} \right)}{\sqrt{(t_1 - t_0)^2 + 4\delta^4}} \right] + \text{erf} \left( \frac{\sqrt{2} \delta \left( t_1 - t_0 \right) k_0 + x_0 - x_1 + \frac{\delta_1}{2}}{\sqrt{(t_1 - t_0)^2 + 4\delta^4}} \right).$$

From Eq. (3.19) and Eq. (2.24) we obtain

$$\phi_1(x, t_2) = \frac{\psi_0(x, t_2)}{2a_1} \left[ -\text{erf}(A_-) + \text{erf}(A_+) \right],$$

where

$$A_- = \frac{(-1)^{1/4} \left( t_1 \left( x - x_1 - \frac{\delta_1}{2} \right) + 2i\delta^2 \left( -x + x_1 + \frac{\delta_1}{2} \right) + t_2 \left( -x_0 - 2i\delta^2 k_0 + x_1 + \frac{\delta_1}{2} \right) \right)}{\sqrt{4\delta^2 + 2it_1 \sqrt{2\delta^2 + it_1 - it_2} \sqrt{t_2}}},$$

and

$$A_+ = \frac{(-1)^{1/4} \left( t_1 \left( x - x_1 + \frac{\delta_1}{2} \right) + 2i\delta^2 \left( -x + x_1 - \frac{\delta_1}{2} \right) + t_2 \left( -x_0 - 2i\delta^2 k_0 + x_1 - \frac{\delta_1}{2} \right) \right)}{\sqrt{4\delta^2 + 2it_1 \sqrt{2\delta^2 + it_1 - it_2} \sqrt{t_2}}}. $$

From Sec. (3.1) we know that the state

$$\psi_1(x, t_2) = \frac{1}{b_1} \left( \psi_0(x, t_2) - a_1 \phi_1(x, t_2) \right).$$
Applying Eq. (3.30) and the above result of \( \phi_1(x, t_2), \psi_1(x, t_2) \) comes out to be

\[
\frac{\psi_0(x, t_2)}{1 - |a_1|^2} \left( 1 - \frac{1}{2} \left[ -erfi(A_-) + erfi(A_+) \right] \right)
\]

The values of \( a_n \) beyond \( a_1 \) get complicated quickly for ordinary integration. For example, in Eq. (3.19), \( |\phi_2, t_2\rangle \) depends on \( \phi_1(x, t_2) \). Therefore \( a_2 = \langle \phi_2, t_2 | \phi_2, t_2 \rangle \) is a nested integral of \( \phi(x, t_2) \) in the range \( x_1 \pm \delta_1/2 \) that includes some function times the error functions as shown above. Meanwhile we want the number of measurements to be high so that Eq. (2.16) is better modeled by Eq. (3.29). Therefore we need the values of all \( a_i, 0 < i \leq n \) for a large \( n \). To address this issue, we make some approximations.

### 3.2 Small detector approximation

We assume the width of the detector \( \delta_1 \) to be very small compared to the width of the wavefunction \( \delta \). We think about this as the detector picking the wavefunction’s amplitude at some point in region \( x_1 \pm \delta_1/2 \) at time \( t_n \). Eq. (3.6) becomes

\[
\lim_{\delta_1 \to 0} |c_1|^2 = |\psi_0(x_1, t_1)|^2 \int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} 1 \, dx
\]

\[
= |\psi_0(x_1, t_1)|^2 \delta_1
\]
3.2 Small detector approximation

which is the area of a rectangle of length $\delta_1$ and height $|\psi_0(x_1,t_1)|^2$ that approximates the

Riemann sum of the values of $|\psi_0(x,t_1)|^2$ in the range $x_1 \pm \frac{\delta_1}{2}$ for very small $\delta_1$. Thus

$$|c_1| = |\psi_0(x_1,t_1)|\sqrt{\delta_1}, \quad (3.31)$$

for very small value of $\delta_1$. The integral in Eq. (3.3) transforms to

$$\begin{align*}
|\phi_1,t\rangle &= \frac{1}{c_1} \langle x_1 | \psi_0,t_1 \rangle |x_1,t_1-t\rangle \int_{x_1-\delta_1/2}^{x_1+\delta_1/2} 1 \, dx \\
&= \frac{1}{c_1} \langle x_1 | \psi_0,t_1 \rangle |x_1,t_1-t\rangle \delta_1.
\end{align*}$$

Using above relation and Eq. (3.12) gives us

$$a_1 = \frac{1}{c_1} \langle \psi_0,t_1 | x_1 \rangle \langle x_1 | \psi_0,t_1 \rangle \delta_1$$

$$= \frac{1}{c_1} |\psi_0(x_1,t_1)|^2 \delta_1$$

$$= \frac{|\psi_0(x_1,t_1)|^2 \delta_1}{|\psi_0(x_1,t_1)|\sqrt{\delta_1}}.$$

Hence, we get

$$a_1 = |\psi_0(x_1,t_1)|\sqrt{\delta_1} = |c_1|. \quad (3.32)$$

Meanwhile, the probability of detecting the particle at time $t_1$ as given by Eq. (3.24) be-

comes

$$P_1(t_1) = |a_1|^2 = |\psi_0(x_1,t_1)|^2 \delta_1 \quad (3.33)$$
Now, from Eq. (3.10), the state $|\psi_1, t\rangle$ approximates to

$$|\psi_1, t\rangle = \frac{1}{b_1} \left( |\psi_0, t\rangle - \frac{a_1}{c_1} x_1 |\psi_0, t_1\rangle x_1, t_1 - t \delta_1 \right)$$

$$= \frac{1}{b_1} \left( |\psi_0, t\rangle - \langle x_1 |\psi_0, t_1\rangle |x_1, t - t_1\rangle \delta_1 \right).$$

Thus, the wavefunction $\psi_1(x, t)$ comes out to be

$$\psi_1(x, t) = \langle x |\psi_1, t \rangle$$

$$= \frac{1}{b_1} \left( \langle x |\psi_0, t \rangle - \langle x_1 |\psi_0, t_1\rangle \langle x, 0 |x_1, t_1 - t \rangle \delta_1 \right)$$

$$= \frac{1}{b_1} \left( \psi_0(x, t) - \psi_0(x_1, t_1) \mathcal{K}(x, t, x_1, t_1) \delta_1 \right).$$

Let us examine $a_2$. Applying Eq. (3.34) and Eq. (3.37) to Eq. (3.23) we get

$$a_2 = \langle \phi_2, t_2 |\psi_1, t_2 \rangle$$

$$= \frac{\delta_1}{c_2} \langle \psi_1, t_2 |x_1 \rangle \langle x_1, 0 \rangle \left( \frac{1}{b_1} (|\psi_0, t_2\rangle - \langle x_1 |\psi_0, t_1\rangle |x_1, t_1 - t_2 \rangle \delta_1) \right)$$

$$= \frac{\delta_1}{c_2 b_1} \left( \langle \psi_1, t_2 |x_1 \rangle \langle x_1, 0 \rangle \psi_0, t_2\rangle - \langle x_1, 0 \rangle \psi_0, t_1\rangle \langle x_1, t_2 |x_1 \rangle \langle x_1, 0 |x_1, t_1 - t_2 \rangle \delta_1 \right)$$

$$= \frac{\delta_1}{c_2 b_1} \psi_1(x_1, t_2) \left( \psi_0(x_1, t_2) - \psi_0(x_1, t_1) \mathcal{K}(x_1, t_2, x_1, t_1) \delta_1 \right)$$

$$= \frac{\delta_1}{c_2} \psi_1(x_1, t_2) |\psi_1(x_1, t_2)|^2$$

$$= \frac{|c_2|^2}{c_2} \delta_1$$

$$= c_2.$$
3.3 Random position approximation

Hence, this approximation is consistent with Sec. (3.1) and does not require any integration. In general, we obtain the following relationships.

\[
|\phi_n(t) = \frac{1}{c_n} (\psi_n - 1, t_n)|x_1(t - t_n)\delta_1, \quad (3.34)
\]

\[
|c_n|^2 = |\psi_n - 1(x_1, t_n)|^2 \delta_1, \quad (3.35)
\]

\[
a_n = |c_n| = |\psi_n - 1(x_1, t_n)| \sqrt{\delta_1}, \quad (3.36)
\]

\[
\psi_n(x, t) = \frac{1}{b_n} \left( \psi_n - 1(x, t) - \psi_n - 1(x_1, t_n) \delta \right). \quad (3.37)
\]

3.3 Random position approximation

The small width approximation where the width of the detector \(\delta_1\) tends to zero is not realistic. This is because a particle detected at an exact location would have infinite momentum to preserve the uncertainty relationship between position and momentum. The Uncertainty principle [1] dictates that the standard deviation in the position of the particle times the standard deviation in its momentum has to be greater than or equal to \(\hbar/2\). Thus, a particle’s position can never be measured with infinite precision. We tackle this issue with a better approximation by putting our detector in random positions within the range \(x_1 \pm \delta_1/2\). First we observe using Riemann sum that

\[
\int_{x_1 - \delta_1/2}^{x_1 + \delta_1/2} |\psi_n - 1(x, t_n)|^2 dx = \lim_{m \to \infty} \sum_{k=1}^{m} |\psi_n - 1(x_k, t_n)|^2 \delta_1 \quad \text{where} \quad -\delta_1/2 \leq x_k \leq \delta_1/2
\]

\[
= |c_{n1}|^2 + |c_{n2}|^2 + |c_{n3}|^2 + \ldots
\]
where the $c_{nk}$ is the value of $c_n$ calculated with Eq. (3.36) but with random $x_i$ in the range $x_1 \pm \frac{\delta_1}{2}$ each time we compute $c_{nk}$. The average value $|c_n|^2$ is the probability of detection at time $t_n$. For some large $m$,

$$
|\bar{c}_n|^2 = \frac{|c_{n1}|^2 + |c_{n2}|^2 + \ldots + |c_{nm}|^2}{m}.
$$

(3.38)

The amplitude of the detected states $\phi_n(x,t_n)$ is given by

$$
a_n = |c_n| = \sqrt{\frac{\sum_{j=1}^{m} |c_{nj}|^2}{m}}.
$$

(3.39)

### 3.4 Results

#### 3.4.1 Comparing the three methods

We calculated the values of $a_1$ using the three different methods above. The actual value $a_{1\alpha}$ was given by Eq. (3.30). We calculated the small detector approximated $a_{1\beta}$ using Eq. (3.32) and the random position approximated $a_{1\gamma}$ using Eq. (3.39) where the superscripts $\alpha, \beta$ and $\gamma$ represent the three methods. We used the following values for our calculation, $k_0 = 20, x_0 = -5, x_1 = -3, \delta = 0.5, n = 1000, t_0 = -0.02, t_f = 0.22,$ and $dt = \frac{t_f - t_0}{n - 1}$. Refering back to the Eq. (3.12), the values of $a_1$ were calculated at time $t_1$ where $t_1 = t_0 + dt$.

One can observe in Fig. (3.1) that the values of $a_1$ calculated by all three methods are close for small $\delta_1$. The small detector approximation deteriorates as $\delta_1$ gets larger. The values of $a_{1\beta}$ are expected to deviate from the actual values. Meanwhile, the random
3.4 Results

Fig. 3.1 Comparison among $a_1^\alpha$ (thick), $a_1^\beta$ (thin) and $a_1^\gamma$ (plot points) as functions of detector width $\delta_1$.

Fig. 3.2 The values $|a_1^\alpha - a_1^\beta|$ shown by a curve superimposed with the values of $|a_1^\alpha - a_1^\gamma|$ shown by the plotted points for various $\delta_1$.

position approximation also deviates from $a_1^\alpha$ as $\delta_1$ gets larger. The mean deviation plotted against $\delta_1$ can be observed in Fig. (3.2) which shows $a_1^\beta$ and $a_1^\gamma$ are equally separated from the mean value $a_1^\alpha$. The methods with which the results were generated are given in
Appendix (1). We now examine the values of $a_2$. We have calculated the values of $a_2$ at time $t_2$ where $t_2 = t_1 + dt$.

Fig. 3.3 Comparison among $a_2^\alpha$ (thick), $a_1^\beta$ (thin) and $a_2^\gamma$ (plot points) as functions of detector width $\delta_1$.

Fig. (3.3) shows larger deviation from $a_2^\alpha$ for the random position approximation model. The $a_2^\alpha$ values were generated using numerical integration method in Mathematica over the integrand $|\psi_1(x,t)|^2dx$ over the range $x_1 \pm \frac{\delta_1}{2}$. Since $a_2^\alpha$ was not calculated with explicit integration, we suspect that the numerical integration underestimated the values of $a_2^\alpha$. The Mathematica code is given in Appendix (2).

We will now examine the small detector approximation and the random position approximation models for various $a_n$. The values of $a_n^\alpha$ for $n > 2$ had nested integrals that are difficult to compute even using the numerical integration method. Therefore $a_n^\beta$ and $a_n^\gamma$ are compared against each other. We plotted the $a_n$ using the small detector approximation and the random position approximation and observed the following results. Both methods
3.4 Results

give similar results for small $\delta_1$ as shown in Fig. (3.4). The corresponding probabilities of
detection versus time are shown by Fig. (3.5),

![Graph](image1.png)

**Fig. 3.4** The values of $a_n^\beta$ and $a_n^\gamma$ as a function of $t_n$ are superimposed. Here, $m = 100$, $\delta_1 = .005$

![Graph](image2.png)

**Fig. 3.5** The probabilities of detection using small detector approximation, $\pi_n^\beta$ are superimposed with the corresponding random position approximation, $\pi_n^\gamma$ values with $m=100$, $\delta_1 = .005$. The time of arrival was calculated to be 0.101(3s.f.) in both cases
For small values of $\delta_1$, the small width approximation and the random position approximation give similar values. However, they start to deviate as $\delta_1$ gets larger as shown by Fig. (3.6). In Fig. (3.6), in the case of for large $\delta_1$, the values of $a_n^\beta$ blows up after some time $t_n$ while $a_n^\gamma$ is well behaved. In this plot we used $m = 10000$ to compensate for the larger detector width $\delta_1 = .035$. This is because as $\delta_1$ gets larger, the integration over the region $x_1 \pm \frac{\delta_1}{2}$ is less likely to be modeled by the random position approximation with a small $m$.

Meanwhile, Fig. (3.7) shows the corresponding probabilities of detection as plotted against time.

So far we have observed that $a_n^\beta$ worked well for small $n$. Meanwhile Fig. (3.6) indicates $a_n^\gamma$ performs better in the case of $a_n$ for larger $n$ values.

The Ehrenfest Theorem [1] allows us to expect the value of $\bar{r}$ to be close to what is predicted classically. From Sec. (2.2), $k_0$ represents the velocity of the free particle for
3.4 Results

Fig. 3.7 The values of $\pi_\beta^n$ (dashed) and $\pi_\gamma^n$ as a function of $t_n$ are superimposed. Here, $m = 10000, \delta_1 = .035$. The time of arrival was calculated to be $0.102 (3 s. f.)$ in both cases.

$\hbar, m = 1$. The time of arrival is expected to be

$$t = \frac{x_1 - x_0}{k_0} = \frac{-3 + 5}{20} = 0.1.$$ 

Hence, we would expect the better model to give the arrival time that is closer to $0.1$. We now study the relationships between the time of arrival and the detection parameters.

3.4.2 Time as a function of detection parameters

In Fig. (3.8) we examine the time of arrival $\bar{t}$ computed using the small detector approximation $\bar{t}_\beta$ and the random position approximation $\bar{t}_\gamma$ as functions of the detector width $\delta_1$. Even with small $m$, the values of $\bar{t}_\gamma$ are more stable and closer to the classical arrival time.

We also plotted the $\bar{t}_\beta$ as the function of the number of measurement $n$. Fig. (3.9) shows
3.4 Results

Fig. 3.8  $\bar{t}_\beta$ and $\bar{t}_\gamma$ (dashed) as a function of $\delta_1$. Line $t = 0.1$ superimposed represents the classical time of arrival.

$\bar{t}_\beta$ for three different values of $\delta_1$.

Fig. 3.9  $\bar{t}_\beta$ as function of $n$ for $\delta_1 = 0.005(l_1)$, $\delta_1 = 0.05(l_2)$ and $\delta_1 = 0.5(l_3)$

In Fig. (3.9), for small $\delta_1 (0.005)$ $\bar{t}_\beta$ is well behaved for $n > 10$. However, for larger $\delta_1 (0.5)$, $\bar{t}_\beta$ highly deviates from the classical value. Here $\bar{t}_\beta > t_f (0.2)$ for the case $\delta_1 (0.5)$ which
shows the limitations of the small width approximation method.

In Fig. (3.10), for medium $\delta_1(0.05)$ $\bar{t}_\gamma$ is well behaved for $n > 10$. However, for larger $\delta_1(0.5)$, $\bar{t}_\gamma$ highly deviates in the beginning. However, with more detection, $\bar{t}_\gamma$ seems to be approaching the mean value 0.1.

It would be interesting to check whether $\bar{t}_\gamma$ for $\delta_1 = 0.5$ gets closer to 0.1 for larger $n$. However, larger values of $n$ makes the computation of $\bar{t}_\gamma$ highly expensive. This arises from the fact that each $a_n$ takes at least $n^2$ operations (ref. Appendix (3)) and is computed $m$ times to give $\bar{a}_n$. Hence, the brute force complexity of $\bar{t}_\gamma$ is at least $n^2m$ operations, which is about $2.5 \times 10^8$ operations for the $n = 500, m = 1000$ case. Some of the values of $\bar{t}$ are shown in Table (3.1).
3.4 Results

Table 3.1 Time of arrival calculated using the small width approximation ($\bar{t}_β$) and the random detector approximation ($\bar{t}_γ$). Here $n = 20$.

<table>
<thead>
<tr>
<th>$δ_1$</th>
<th>$\bar{t}_β$</th>
<th>$\bar{t}_γ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.100826</td>
<td>0.100818 (m=100)</td>
</tr>
<tr>
<td>0.035</td>
<td>0.102417</td>
<td>0.102283 (m=10000)</td>
</tr>
</tbody>
</table>
Chapter 4

A Non-Directional Dynamic Approach

There is no reason that our assumptions regarding time being a directional and non-dynamic variable should hold true. Firstly, these assumptions come at the expense of the treatment of time on equal footing with space. For example, a state \( \psi(x,t) \) can be measured at position \( x_n < x_{n-1} \) for any \( n \). This is because we are free to move in the one-dimensional space in both directions. Hence, we do not put directionality constraint on the position observable.

Similarly, the position variable is dynamic such that for a normalized wavefunction the \( n^{th} \) detection has a non-zero probability at position anywhere within the open interval \( (-\infty, \infty) \).

Secondly, the restrictions related to directionality and non-dynamism of the time variable do not account for specific instances in Quantum Mechanics. For example, the ambiguous explanations of the delayed choice experiment [11] and the quantum eraser [12] make us question the validity of the directionality and non-dynamism constraints even further. In order to treat time as equal with a position variable and not just a mere parameter we let go of the two constraints. We will now deal with time variable that is non-directional
4.1 Equal treatment of Time and Space

For this section we will deal with the small detector approximation with $\delta_1 \leq 0.005$. The computational complexities of random detector approximations along with this approach will be discussed later.

Our approach in Sec. (3.1.1) will hold except that the first measurement now may occur at any time $t_i \leq t \leq t_f$. Similarly, the second measurement may occur at time $t < t_1$ or $t > t_2$. Hence, under this approach we can combine Eq. (3.7) with Eq. (3.16) to get

$$|\psi_0, t\rangle = a_1 |\phi_1, t\rangle + b_1 \left( a_2 |\phi_2, t\rangle + b_2 |\psi_2, t\rangle \right),$$

where $|\psi_2, t\rangle$ and the later states are given by Eq. (3.26). Thus, $|\psi_0, t\rangle$ can be written as the superposition of all possible detection states

$$|\psi_0, t\rangle = a_1 |\phi_1, t\rangle + b_1 \left( a_2 |\phi_2, t\rangle + b_2 (a_3 |\phi_3, t\rangle + ...) \right)$$

Here $a_n^2$ is the probability of detecting the state $|\phi_n, t\rangle$ at $t_i \leq t < t_f$ given it is not detected at the state $|\phi_{n-1}, t\rangle$. The true probability of detecting the particle in the state $|\phi_n, t\rangle$ is given by Eq. (3.25). For example,

$$\pi_2(t) = P_2(t) \left( 1 - P_1(t) \right)$$
4.1 Equal treatment of Time and Space

\[ a_2(t) = a^2(t) b_1^2(t). \]

which is just the square of the coefficient of the state \( \phi_2, t \) in Eq. (4.2). The probability of detection at any time \( T_n \) is the square of the coefficient of each detected state in Eq. (4.2) given by

\[ \pi_n(T_n) = \prod_{i=1}^{n-1} b_i^2(T_i) a_n^2(T_n), \quad (4.3) \]

which is analogous to Eq. (3.28) except that under this approach, \( t_i \leq T_n < t_f \). Hence, there is a non-zero probability of finding a particle at any eigenstate within the detection period which reflects the dynamic nature of the time variable. For time \( T_i \), the probability density is

\[ \rho(T_i) = \frac{\sum_{j=0}^{n} \pi_j(T_i)}{\sum_{i=1}^{k} \sum_{j=0}^{n} \pi_j(T_i)}, \]

where the denominator is the normalization constant such that the sum over the detection span \( t_i \leq t < t_f \) is 1. i.e.

\[ \sum_{i=1}^{k} \rho(T_i) = \frac{\sum_{i=1}^{k} \sum_{j=0}^{n} \pi_j(T_i)}{\sum_{i=1}^{k} \sum_{j=0}^{n} \pi_j(T_i)} = 1. \]

Finally, the time of arrival is the probability of getting \( T_n \) where \( t_0 \leq T_i < t_f \), which is

\[ \bar{T} = \frac{\sum_{i=1}^{k} \sum_{j=0}^{n} T_i \pi_j(T_i)}{\sum_{i=1}^{k} \sum_{j=0}^{n} \pi_j(T_i)}, \quad (4.4) \]
where \( k \) is the number of steps into which we divide the detection period \( t_i - t_f \) and \( n \) is the number of measurements. Previously, \( n \) and \( k \) were the same.

Let \( n = 4 \) and \( k = 10 \). The possible combinations of \( T_i, 0 < i \leq n \) includes \{\( t_1, t_2, t_3, t_4 \)\}, \{\( t_2, t_4, t_6, t_8 \)\}, \{\( t_7, t_4, t_2, t_9 \)\}, \{\( t_7, t_7, t_2, t_9 \)\}, \{\( t_7, t_4, t_2, t_4 \)\}, etc. The probability of being detected in the state \( \psi_0(x,t) \) at time \( T_1 \) during the first measurement is

\[
|a_1(T_1)|^2 = |\psi_0(x_d, T_1)|^2 \delta_1. \tag{4.5}
\]

The probability of not being detected in the state \( \psi_0(x,t) \) is \(|b_1(T_1)|^2 = 1 - |a_1(T_1)|^2\). Hence, Eq. (3.7) becomes

\[
\psi_0(x,t) = a_1(T_1)\phi_1(x,t) + b_1(T_1)\psi_1(x,t),
\]

where \( T_1 \) can have any value in the range \( t_0 - t_f \) and is not to be confused with \( t_1 \). The subscript \( n \) in \( T_n \) is kept to keep track of the appropriate detected state \( \phi_n(x,t) \). Additionally, \( a_1 \) is a function of time \( t_0 \leq T_1 < t_f \). This represents the non-directionality of the time variable. In previous sections, we computed \( a_1 \) at time \( t_1 \), \( a_2 \) at time \( t_2 \) and so on. The influence of the past events on the present was assumed to be null. For example, in Sec. (3.1), the lifetime of state \( |\phi_n, t\rangle \) was \( t_{n-1} \leq t < t_n \). If we were at time \( t_3 \), for instance, and we made a detection, then \( a_n \) for \( n < 3 \) must be 0 at time \( t_3 \). This is because after each measurement at time \( t_n \) for \( n < 3 \), we had assumed that the wavefunction collapsed to its subsequent non-detected states, \( \psi_1(x,t) \) at time \( t_1 \) and and \( \psi_2(x,t) \) at time \( t_2 \). Thus
4.1 Equal treatment of Time and Space

$a_1(t_3) = a_2(t_3) = 0$. In general we had assumed that $a_n(t_m) = 0$ for all $m > n$. We now allow $a_n(t_m)$ to have a non zero value for $m > n$. Some modifications from the Sec. (3.1) are given below.

\[
|\phi_n(t)\rangle = \frac{1}{c_n} \langle x|\psi_{n-1}, T_n\rangle |x_d, T_n - t\rangle \delta_1, \tag{4.6}
\]

which gives,

\[
\phi_n(x,t) = \langle x|\phi_n(t)\rangle = \frac{1}{c_n} \psi_{n-1}(x,T_n) \langle x,0|x_d, T_n - t\rangle \delta_1 = \frac{1}{c_n} \psi_0(x_d, T_n) \mathcal{K}(x,t,x_d,T_n) \delta_1.
\]

Similarly, Eq. (3.37) becomes

\[
\psi_n(x,t) = \frac{1}{b_1} \left[ \psi_{n-1}(x,t) - \psi_{n-1}(x_d,T_1) \mathcal{K}(x,t,x_d,T_1) \delta_1 \right]. \tag{4.7}
\]

### 4.1.1 Adjacent repetitions

Suppose a particle is not detected during the first measurement. In such a case, the wavefunction momentarily collapses to the normalized state $\psi_1(x,t)$. In the case $T_2 = T_1$ i.e. we make the next measurement at the same time we made the first measurement

\[
|a_2|^2 = |\psi_1(x_d, T_2)|^2 \delta_1 = |\psi_1(x_d, T_1)|^2 \delta_1
\]
\[ \psi_n(x, t) = \frac{1}{b_n} \left( \psi_{n-1}(x, t) - a_1 \phi_n(x, t) \right) \]

\[ = \frac{1}{b_n} \left( \psi_{n-1}(x, t) - \int_{x_d - \delta_1/2}^{x_d + \delta_1/2} \psi_{n-1}(x', T_n) \langle x, 0 | x', T_n - t \rangle dx' \right), \]

which comes from the definition of \( \phi_n(x, t) \) in Sec. (3.1). Now,

\[ \psi_n(x, T_n) = \frac{1}{b_n} \left( \psi_{n-1}(x, T_n) - \int_{x_d - \delta_1/2}^{x_d + \delta_1/2} \psi_{n-1}(x', T_n) \langle x, 0 | x', 0 \rangle dx' \right) \]

\[ = \frac{1}{b_n} \left( \psi_{n-1}(x, T_n) - \int_{x_d - \delta_1/2}^{x_d + \delta_1/2} \psi_{n-1}(x', T_n) \delta(x - x') dx' \right) \]

\[ = \begin{cases} 
\frac{1}{b_n} \left( \psi_{n-1}(x, T_n) - \psi_{n-1}(x, T_n) \right) & x_d - \frac{\delta_1}{2} \leq x \leq x_d + \frac{\delta_1}{2} \\
\frac{1}{b_n} \left( \psi_{n-1}(x, T_n) - 0 \right) & \text{otherwise.} 
\end{cases} \]

Thus, we have

\[ \psi_n(x, T_n) = \begin{cases} 
0 & x_d - \frac{\delta_1}{2} \leq x \leq x_d + \frac{\delta_1}{2} \\
\frac{1}{b_n} \psi_{n-1}(x, T_n) & \text{otherwise.} 
\end{cases} \]

\[ 4.1 \text{ Equal treatment of Time and Space} \]

\[ = 0. \]

i.e. there is no probability of detecting the particle during this second measurement. This is because
4.1 Equal treatment of Time and Space

The above results stem from the orthogonality of $\phi_n(x, T_n)$ and $\psi_n(x, T_n)$. If the particle was detected at time $T_n$, then it will be in state $\phi_n(x, T_n)$ momentarily. Since, this state occurs in the region $x_d \pm \frac{\delta_1}{2}$, $\psi_n(x, T_n) = \psi_{n-1}(x, T_n)$ is 0. Meanwhile, if the particle is not detected, then the particle is in the non-detected state at time $T_n$ which means that the amplitude of particle in the range $x_d \pm \frac{\delta_1}{2}$ is 0, i.e., $a_n(T_n) = 0$ and $b_n(T_n) = 1$. The fact that $\psi_n(x, T_n) = \psi_{n-1}(x, T_n)$ also means that the particle will not change its state at time $T_n$. For any $T_n$,

$$T_n = T_{n-1} \implies a_n = 0 \implies \pi_n(T_n) = 0. \quad (4.8)$$

4.1.2 Non-adjacent repetitions

If $T_n \neq T_{n-1}$, then

$$|a_n|^2 = |\psi_{n-1}(x_d, T_2)^2 \delta_1| \neq 0. \quad (4.9)$$

Let’s examine the case $T_3 = T_1$ given $T_2 \neq T_1$. From Eq. (4.9), we have

$$a_3^2 = |\psi_2(x_d, T_2)^2 \delta_1|,$$

where $\psi_2(x_d, T_2)$ can be deduced from Eq. (4.7). We get

$$\psi_2(x_d, T_1) = \frac{1}{b_2(T_1)} \left( \psi_1(x_d, T_1) - \psi_1(x_d, T_2) \cdot \mathcal{K}(x_d, T_1, x_d, T_2) \delta_1 \right).$$
4.1 Equal treatment of Time and Space

Now, the term $\psi_1(x_d,T_1)$ also stems from Eq. (4.7) as

$$\psi_1(x_d,T_1) = \frac{1}{b_1(T_1)} \left( \psi_0(x_d,T_1) - \psi_1(x_d,T_1) \phi(x_d,0,x_d,T_1-T_1) \delta_1 \right)$$

$$= \frac{1}{b_1(T_1)} \left( \psi_0(x_d,T_1) - \psi_1(x_d,T_1) \right)$$

$$= 0.$$

Therefore,

$$\psi_2(x_d,T_1) = \frac{1}{b_2'(T_1)} \left( \psi_1(x_d,T_2) \phi(x_d,T_1,x_d,T_2) \delta_1 \right),$$

where $b_2'(T_1)$ is the new normalization constant. In any case, $\psi_2(x_d,T_1)$ is non zero. Hence, $a_3(T_1) \neq 0$. The probability of detection is given by

$$\pi_3(T_1) = |b_1(T_1)|^2 |b_2(T_2)|^2 |a_3(T_1)|^2$$

$$\neq |a_1(T_1)|^2 = \pi_1(T_1).$$

even though the particle was detected at time $T_1$.

The result can be interpreted as such: $\pi_1(T_1)$ is the probability that we make one measurement at time $T_1$ and detect the particle. There will be no more evolution of the wavefunction as it permanently chooses a detected state. However, if we do not detect the particle, the wavefunction momentarily collapses into a non-detected state. A non-detected state such as $\psi_1(x,t)$ has zero amplitude in the range $x_d \pm \delta_1/2$ at time $T_1$. However, the
wavefunction soon starts spreading out again, in accordance with the Schrödinger equation and recovers the probability of being detected in the range \( x_d \pm \delta_1/2 \). We then make another measurement at time \( T_2 \). Here \( T_2 \) may be earlier than \( T_1 \), equal to \( T_1 \) or later than \( T_1 \).

In the case when \( T_2 = T_1 \), there is no probability that the particle will be detected with this measurement because the wavefunction has not evolved its state.

In the case \( T_2 > T_2 \) or \( T_2 < T_1 \) (assuming we can make measurements this way) the wavefunction momentarily collapses exclusively to state \( \phi_2(x, T_2) \) or \( \psi_2(x, T_2) \) depending on whether we detect the particle or not. Let’s assume that we fail to detect the particle at \( T_2 \) and detect it on the third measurement at time \( T_3 \). If \( T_3 = T_1 \), we detect the state \( \phi_3(x, t) \neq \phi_1(x, t) \). This is because although the Schrödinger equation is time symmetric [1], we made a measurement in between, causing an irreversible collapse at time \( T_2 \).

### 4.2 Results

Table (4.1) shows the time of arrival for various values of \( m \), the number of steps into which we divide the time \( t_0 \leq t < t_f \) and \( n \), the number of measurements. Small values of \( n \) were used due to the computational complexity of this method. For example, a case when \( m=100 \) and \( n = 3 \) has to go through \( 100^3 \) computations at least twice to give the time of arrival. This is because each value of \( T_n \) has 100 choices of \( t_n \) when we compute the amplitudes \( a_n \). We then have to compute \( \pi_n \) for all \( T_n \) which has the same efficiency. In general, the brute force computation of \( \bar{r} \) has efficiency of least \( 2m^n \) operations. Hence, the dynamic and non-directional approach of calculating \( \bar{r} \) is computationally expensive. The methods
4.2 Results of calculating $\bar{t}$ with this approach can be examined in Appendix (3).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\bar{t}$</th>
<th>$\bar{t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>0.100528</td>
<td>0.100575</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>0.100532</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>0.1005076</td>
<td>0.100522</td>
</tr>
<tr>
<td>100</td>
<td>3</td>
<td>0.100506</td>
<td>0.100515</td>
</tr>
</tbody>
</table>

Table 4.1 The non-directional and dynamic $\bar{t}$ and the directional and dynamic $\bar{t}$ with measurement parameters. Here $\delta_1 = 0.005$

Meanwhile, even for the small values of $n$ and $m$, we obtain results that are closer to classical arrival time of the particle than the non-dynamic and directional approach. We can compare a value in Table (4.1) obtained for $\delta_1 = 0.005$ and $n = m = 7$ with $\bar{t}_2$ for $n = 7$ using the small width approximation. Using $n = 7$ in the small detector approximation model we get 0.100817 for the time of arrival. Our result with this approach is definitely closer to the classical prediction.

The fourth column in Table (4.1) represents the time variable that is dynamic and directional. Here, we only used the $T_n$ such that $T_{n+1} \geq Tn$ for all $n$. For example, if $n = 2$ and $m = 3$, the tuples of $\{T_1, T_2\} = \{t_1, t_1\}, \{t_1, t_2\}, \{t_1, t_3\}, \{t_2, t_2\}, \{t_2, t_3\}, \{t_3, t_3\}$. Although this method gives better results than the previous sections, the time of arrival that is non-directional and dynamic is still closer to 0.1.
4.3 Discussion

One of the consequences of this formalism is that the probabilities of detecting the particle at a precise space-time location are somehow different. Here, $\pi_3(T_1)$ is the probability of detecting the particle at state $\psi_2(x_d, T_1)$ while $\pi_1(T_1)$ is the probability of detecting the particle at state $\psi_0(x_d, T_1)$. Let $T_1 = T_3 = t_1$. If we detect the particle in the state $\psi_2(x_d, T_1)$, then the detected state is $\phi_3(x_d, T_1)$. In the traditional Copenhagen interpretation, where time is an independent parameter of which the other dynamical quantities such as position and momentum are functions, the above result would be nonsensical. The traditional Copenhagen interpretation of time is non dynamic and directional with respect to measurements. Under such a framework, when we make a detection $a_3(t_1)$, we detect a state that can only exist in the interval $t_2 \leq t < t_3$. The state $|\psi_2\rangle$ is caused by two measurements on the system. Somehow the wavefunction changes its state from $|\psi_0\rangle$ to $|\psi_2\rangle$ before we detect it at time $t_1$, the time of the first measurement.

Dynamic and non-directional treatment of the time can provide a better explanation. In this approach, the initial wavefunction $\psi_0(x, t)$ is in the superposition of all the detected states at all times as in Eq. (4.2). When we make a measurement at time $T_1 = t_1$, we can detect the wavefunction at any state. For example, we may detect the state $|\phi_3, t_1\rangle$ or even $|\phi_{100}, t_1\rangle$ which have different probabilities than the state $|\phi_1, t_1\rangle$. However, we expect the probability of detecting the state $|\phi_{100}, t\rangle$ at time $t_1$ to be very small. One way to examine this would be to plot the values of $a_n$ against all $t_0 \leq t < t_f$. We expect the amplitude of $a_n$ to peak at time $t_n$. For example, $a_5$ is expected to have a high values around the time $t_5$.
4.3 Discussion

with some spread on either side. Similarly, one may also plot $\pi_n$ against $t_n$ and examine the results.

However, $a_n$ having non-zero values at all times has interesting consequences. For example, when we detect the particle at time $t_1$, we could have detected any state $|\phi_n, t\rangle$. Detecting $|\phi_3, t_1\rangle$ implies that the total of three measurements were performed and each measurement other than at $t_1$ could have happened sometime in the past or in the distant future. We would not know because we detect the particle at time $t_1$. Meanwhile, the measurements of the future and past must have influenced our detection because the probabilities are different. Amidst such nonintuitive consequences we acknowledge the famous delayed choice experiment where somehow future measurements affect the history of the particle [11]. Further exploration needs to be done on the relationship between the dynamic and non-directional time variable with such experiments where the parametric treatment of time fails to provide a satisfactory explanation.
Chapter 5

Summary

In this project we devised mechanisms to calculate the time of arrival for a free particle wavefunction. In doing so, we made initial assumptions that the time operator is directional and non-dynamic. We took into account the effects of measurements. We examined the detected and non-detected states and the probabilities associated with them. We then used two approximations, small width and the random position to compute the time of arrival of the particle. We then compared the results of the two methods with the classical arrival time.

In the later part of the project we focused on the time of arrival using a dynamic and non-directional time variable. In doing so we modified our time of arrival definition to accommodate for the probability densities that were unaccounted for in the previous chapters. We then computed the time of arrival under the dynamic and non-directional approach and came up with values that were closer to the classical predictions. Specifically, we treated time on more equal footing with position than the traditional quantum mechanics. Finally, we discussed some interesting and counterintuitive consequences of this formalism.