Quantum Time: Formalism and Applications

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by

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Abstract

Treating time on an equal footing with space in quantum mechanics is a natural desire motivated by the similar concept in special relativity. We believe that this concept can shed light on ambiguous problems in quantum mechanics. In this thesis, we develop a new quantum formalism in which time is a dynamic variable and study its effect on some applications in quantum mechanics. We show that treating space and time equally not only effectively solves the problem of arrival time, but also greatly enhances our understanding of quantum mechanics in general.
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1 Introduction

In traditional quantum mechanics, whereas position and momentum are treated as quantum observables represented by self-adjoint operators, time is treated as a c-number. The famous Pauli’s theorem [1] asserts that the existence of a self-adjoint time operator canonically conjugate with a Hamiltonian implies that the time operator and the Hamiltonian have continuous spectra spanning from $-\infty$ to $\infty$, which is incompatible with the semi-bounded nature of the Hamiltonian spectrum. Thus Pauli concluded that time in quantum mechanics should only be regarded as an ordinary number, not a dynamical variable.

However, the lack of dynamical significance of time has led to unsatisfactory answers to many questions, such as ambiguous interpretations for the time-energy uncertainty relation and the arrival time problem, which is a problem of finding the probability of detecting a particle at a certain time $t$. Moreover, the idea that space and time should be treated on an equal footing is one of the hallmarks of relativity. For example, in quantum field theory, time plays a two-fold role as a parameter and an operator through the four-dimensional angular momentum tensor operator [2]. For these reasons, there has been continued work on forming a new quantum formalism that treats time as a dynamical variable.

Still, one has to overcome the argument that Pauli has raised in order to develop this new formalism. As Muga [3] noted, currently there are two kinds of solutions to this problem. The first one strives to build an ideal vector space outside the standard framework of quantum mechanics to treat space and time equally. However, this approach is of no interest to us, since techniques that we often use in quantum mechanics are inapplicable here. The second approach is to construct self-adjoint time operators from the canonical commutation relation.

We take a route similar to the second method to find a formalism that treats time as a dynamical variable. In the first part of this paper, we extend the theory of Hamiltonian mechanics to a formalism that treats time and space equally. We use this formalism to find Poisson brackets that give rise to commutation relations, through which we derive a new
spacetime representation and derive the space and time translational operators. In the second part, we look at several applications in quantum mechanics to see how this formalism helps forge better understanding in application. First, we derive Schrödinger’s equation and the spacetime propagator as natural by-products of the spacetime representation, and examine how spacetime behaves under the translational operator. Second, we deduce the energy-time uncertainty relation. In the last application, we provide a solution for the problem of time of arrival in quantum mechanics.
2 Extended Lagrangian and Hamiltonian Mechanics

In this section, we develop an extended theory of Lagrangian and Hamiltonian mechanics that treats time and spacial coordinates both as dynamical variables. We start by introducing generalized coordinates, in terms of which we rewrite the classical Lagrange equations and Hamiltonian mechanics, and then discuss the Poisson Brackets.

2.1 Generalized Coordinates

A mechanical system can be fully specified by a set of $D$ independent variables, $q_i = q_1, q_2, \ldots, q_D$, which is a set of generalized coordinates. The momentum $p_i$ of each variable is calculated from the Lagrangian $L(q, q', t)$ as

$$p_i = p_i(q, q', t) = \frac{\partial L(q, q', t)}{\partial q'_i},$$

(2.1)

where $q'$ is the derivative of $q$ with respect to time.

In the generalized coordinates, a path is specified parametrically. This is called the General Parametric Method [4], under which all coordinates, including time, are parametrized by a monotonically varying parameter $\beta$,

$$q_\mu = q_\mu(\beta), \quad p_\mu = p_\mu(\beta),$$

(2.2)

where the subscript $\mu$ runs from 0 to D, as distinguished from subscript $i$ which runs from 1 to D. The zeroth component is defined as time multiplied by the speed of light $c$, or $q_0 = ct$, to ensure that all $q_\mu$’s have the same dimension. For simplicity, we take $c = 1$; thus $q_0 = t$.

In this paper, we will denote the derivative with respect to $\beta$ as a dot over a quantity, to make a distinction from the derivative with respect to time, which is denoted by the prime
notation,

\[ \dot{q}_i = \frac{dq_i}{d\beta}, \quad q'_i = \frac{dq_i}{dt}. \quad (2.3) \]

The parameter \( \beta \) is initially unspecified. It will only be specified after all partial derivatives have been taken and the final differential equations of motion have been obtained. It is usually chosen such that the differential equations of motion are as simple as possible. In special relativity, \( \beta \) is often defined as the proper time.

### 2.2 Extended Lagrangian

The extended Lagrangian can be obtained from Hamilton's principle, which states that the natural path of motion of a system makes the action integral an extremum. The action integral is defined as

\[ I = \int_{t_1}^{t_2} L (q, q', t) \, dt. \quad (2.4) \]

In terms of the \( \beta \)-parametrized coordinates which include time as a coordinate, the traditional Lagrangian can be rewritten as

\[ L = L (q, q', t) = L (q_\mu, q'_i), \]

where \( q_\mu \) includes the time coordinate \( q_0 \). We now obtain the new action integral with the generalized coordinates \( q_\mu \) as follows. Using the fact that \( q_0 = t \) and applying the chain rule to Eq. (2.4), we get

\[ I = \int_{t_1}^{t_2} L (q_\mu, q'_i) \, dt = \int_{\beta_1}^{\beta_2} L \left( q_\mu, \frac{\dot{q}_i}{q_0} \right) \dot{q}_0 \, d\beta \equiv \int_{\beta_1}^{\beta_2} \mathcal{L} (q_\mu, \dot{q}_\mu) \, d\beta. \quad (2.5) \]
Thus we define the extended Lagrangian of the generalized coordinates as

\[ L(q_\mu, \dot{q}_\mu) = \dot{q}_0 L(q_\mu, \frac{\dot{q}_\mu}{\dot{q}_0}). \]  

(2.6)

### 2.3 Generalized Momenta

The generalized momenta is defined as

\[ p_\nu = p_\nu(q_\mu, \dot{q}_\mu) = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_\nu}, \]  

(2.7)

where \( \nu \) is to differentiate from \( \mu \) and runs from 0 to D. This form is similar to the traditional Lagrangian theory, except that \( \dot{q}_\mu \) now denotes differentiation with respect to \( \beta \), and an additional momentum \( p_0 \) is included.

The additional zeroth component of the generalized momenta, \( p_0 \), can be obtained by applying Eq. (2.7) to Eq. (2.6),

\[
p_0(q_\mu, \dot{q}_\mu) = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} = L(q_\mu, q'_\mu) + \dot{q}_0 \sum_{j=1}^{D} \frac{\partial L(q_\mu, q'_j)}{\partial q'_j} \frac{\dot{q}_j}{\dot{q}_0} \\
= L(q_\mu, q'_\mu) - \dot{q}_0 \sum_{j=1}^{D} p_j(q_\mu, q'_j) \frac{\dot{q}_j}{\dot{q}_0} \\
= L(q_\mu, q'_\mu) - \sum_{j=1}^{D} p_j(q_\mu, q'_j) q'_j \\
\equiv -H(q_\mu, q'_i),
\]

(2.8)

where \( H(q_\mu, q'_i) \) is the traditional generalized energy function rewritten in the generalized notation \([4, 37]\). Summarizing the above equation, we have

\[
\frac{\partial L(q_\mu, \dot{q}_i)}{\partial \dot{q}_0} = p_0(q_\mu, \dot{q}_i) = -H(q_\mu, q'_i).
\]

(2.9)
As is customary, we express the dependency among the extended phase-space variables $q, p$ by defining a dependency function $K(q, p)$ [5],

$$K(q, p) = p_0(q_{\mu}, \dot{q}_\mu) + H(q_{\mu}, p_i(q_i, \dot{q}_i)). \tag{2.10}$$

It has been proved that the dependency functions has to be zero for the case where functions are functionally dependent [4, 563] [6, 424]. This is indeed the case here: as Eq. (2.8), $p_0(q_{\mu}, \dot{q}_i) = -H(q_{\mu}, q'_i)$, hence $K(q, p) = 0$. We will see this dependency relation recovered again in the extended Hamiltonian mechanics.

### 2.4 Extended Lagrange Equations

Applying Hamilton’s Principle to the extended Lagrangian of Eq. (2.6), we get

$$\frac{\partial L(q_{\nu}, \dot{q}_\nu)}{\partial q_\mu} - \frac{d}{d\beta} \left( \frac{\partial L(q_{\nu}, \dot{q}_\nu)}{\partial \dot{q}_\mu} \right) = 0. \tag{2.11}$$

This is the extended Lagrange equations. We now show that the extended Lagrange equations are equivalent to the traditional Lagrange equations for $\mu \neq 0$, and is equivalent to the generalized energy theorem for $i = 0$. First we look at the case of $\mu \neq 0$. Our strategy is to show that we can arrive at the extended Lagrange equations from the traditional Lagrange equations. The traditional Lagrange equations are

$$\frac{\partial L(q_{\nu}, q'_i)}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial L(q_{\nu}, q'_i)}{\partial q'_j} \right) = 0, \tag{2.12}$$

where index $j$ is to differentiate from $i$ and runs from 1 to D. Multiply both sides by $\dot{q}_0$, which is $\frac{dq_0}{d\beta}$, we get

$$\dot{q}_0 \frac{\partial L(q_{\nu}, q'_i)}{\partial q_j} - \frac{d}{d\beta} \left( \frac{\partial L(q_{\nu}, q'_i)}{\partial q'_j} \right) = 0. \tag{2.13}$$

This is equivalent to Eq. (2.11) because, using $q_0 = t$, the first term at the left hand side
becomes
\[ \dot{q}_0 \frac{\partial L(q_\nu, q'_i)}{\partial q_0} = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial q_0}, \]
and for the second term
\[ \frac{\partial L(q_\mu, q'_j)}{\partial q'_i} = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_i}, \]
using Eq. (2.8). Thus both terms are equivalent to the extended form.

Similarly, for \( \mu = 0 \), we show that we can arrive at the extended Lagrange equations from the generalized energy theorem, which is
\[ \frac{\partial H(q_\nu, q'_i)}{\partial t} = -\frac{\partial L(q_\nu, q'_i)}{\partial t}. \] (2.14)
in terms of the Lagrangian [4]. Here \( H \) is the generalized energy function defined in Eq. (2.9).

Multiplying both sides by \( \dot{q}_0 \) and change \( t \) to \( q_0 \),
\[ \frac{\partial H(q_\nu, q'_i)}{\partial \beta} = -\dot{q}_0 \frac{\partial L(q_\nu, q'_i)}{\partial q_0}. \]

Applying Eq. (2.9) to the left hand side and simplify the right hand side, we have
\[ \frac{d}{d\beta} \left( -\frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} \right) = -\frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial q_0}, \]
which is equivalent to the \( \mu = 0 \) component of Eq. (2.11). We will see that this combination of the traditional theory and the generalized energy theorem occurs again in the extended Hamiltonian Mechanics.

### 2.5 Extended Hamiltonian Mechanics

Hamiltonian's theory writes the equations of motion in terms of a set of phase-space variables \( q \) and conjugate momentum \( p \). The variables \( q, p \) are called the canonical coordinates.
For the extended Hamiltonian theory, the set of canonical coordinates is expanded to include new coordinates \( q_0 = t \) and its momentum \( p_0 \), so

\[
q, p = q_0, q_1, \ldots, q_D, p_0, p_1, \ldots, p_D,
\]

which makes the phase space \((2D+2)\) dimensional.

Analogous to the extended Lagrangian theory, the extended Hamiltonian theory can be obtained by converting the traditional Hamilton equations to a parametric form with respect to \( \beta \). It is important to note that in the Lagrangian theory, \( p_\mu \) are derived quantities; in the Hamiltonian theory, however, they are independent canonical coordinates in phase space. Thus, Eq. (2.9) is not necessarily true, and we need to re-determine the value of the dependency function \( \mathcal{K}(q, p) \) from Hamilton’s equations. Let the dependency function \( \mathcal{K}(q_\mu, p_\mu) = p_0(q_\mu, \dot{q}_\mu) + H(q_\mu, p_i) \) be the extended Hamiltonian. In analogy to the classical forms, we write the extended Hamilton equations as

\[
\dot{q}_\mu = \frac{\partial \mathcal{K}(q_\mu, p_\mu)}{\partial p_\mu}, \quad \dot{p}_\mu = -\frac{\partial \mathcal{K}(q_\mu, p_\mu)}{\partial q_\mu}.
\]

(2.15)

An immediate consequence of these equations is that

\[
\frac{d\mathcal{K}}{d\beta} = \sum_{\mu=0}^{D} \left( \frac{\partial \mathcal{K}(q_\mu, p_\mu)}{\partial q_\mu} \dot{q}_\mu + \frac{\partial \mathcal{K}(q_\mu, p_\mu)}{\partial p_\mu} \dot{p}_\mu \right) = \sum_{\mu=0}^{D} (-\dot{p}_\mu \dot{q}_\mu + \dot{p}_\mu \dot{q}_\mu) = 0.
\]

As only the derivatives of \( \mathcal{K} \) are going to be of our interest, \( \mathcal{K} \) could be chosen as any constant. For the sake of simplicity, we choose \( \mathcal{K} = 0 \). Thus the dependency relation \( \mathcal{K} = p_0 + H = 0 \) is recovered as a natural consequence of the Hamilton equations of motion.

We now prove that the extended Hamilton equations are a combination of the traditional
Hamilton equations,

\[ q_i' = \frac{\partial H(q_\mu, p_i)}{\partial p_i}, \quad p_i' = -\frac{\partial H(q_\mu, p_i)}{\partial q_i} \]  

(2.16)

when \( i \neq 0 \), and the traditional Hamiltonian form of the generalized energy theorem

\[ \frac{dH(q_\mu, p_i)}{dt} = \frac{\partial H(q_\mu, p_i)}{\partial q_0} \]  

(2.17)

when \( i = 0 \).

First we look at the \( i = 0 \) case. Substituting \( K = p_0(q_\mu, \dot{q}_i) + H(q_\mu, p_i) \) into the first equation of Eq. (2.15), we obtain

\[ \dot{q}_0 = \frac{\partial(p_0(q_\mu, \dot{q}_i) + H(q_\mu, p_i))}{\partial p_0} = 1. \]  

(2.18)

On the other hand, by definition

\[ \dot{q}_0 = \frac{dq_0}{d\beta} q_0' = q_0'. \]

Equating this with Eq. (2.18), we get \( \dot{q}_0 = \frac{dq_0}{d\beta} q_0' = q_0' = 1 \). Similarly, substituting \( K = p_0(q_\mu, \dot{q}_i) + H(q_\mu, p_i) \) into the second equation of Eq. (2.15), we have

\[ \dot{p}_0 = -\frac{\partial(p_0(q_\mu, \dot{q}_i) + H(q_\mu, p_i))}{\partial q_0} = -\frac{\partial H(q_\mu, p_i)}{\partial q_0}. \]  

(2.19)

By definition, however, \( \dot{p}_0 = \frac{dq_0}{d\beta} p_0' \). Since \( \frac{dq_0}{d\beta} = 1 \), we then have

\[ \dot{p}_0 = \frac{dq_0}{d\beta} p_0' = p_0' = -\frac{\partial H(q_\mu, p_i)}{\partial q_0} = -\frac{\partial H(q_\mu, p_i)}{\partial t}. \]

Hence the Eq. (2.17) is recovered, as was to be proved.

Now we look at the \( i \neq 0 \) case. On the one hand, substituting \( K = p_0 + H \) into the
extended Hamiltonian equations of Eq. (2.15) gives
\begin{align}
\dot{q}_i &= \frac{\partial H(q_\mu, p_i)}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H(q_\mu, p_i)}{\partial q_i}.
\end{align}
(2.20)

By definition of the dot notation and the fact that \( \dot{q}_0 = \frac{dq_0}{d\beta} = 1 \), we have
\begin{align}
\dot{q}_i &= \frac{dt}{d\beta} q'_i = q'_i, \\
\dot{p}_i &= \frac{dt}{d\beta} p'_i = p'_i.
\end{align}
Equating them with Eq. (2.20), we recover the traditional Hamilton equations in Eq. (2.16).

2.6 Extended Poisson Brackets

In the extended Hamiltonian mechanics, all quantities are assumed to be functions of phase space variables. The derivative of the phase-space function \( f(q_\nu, p_\nu) \) with respect to \( \beta \) can be calculated as the following [4],
\begin{align}
\dot{f}(q_\nu, p_\nu) &= \frac{df}{d\beta} = \sum_{\mu=0}^{D} \left( \frac{\partial f(q_\nu, p_\nu)}{\partial q_\mu} \dot{q}_\mu + \frac{\partial f(q_\nu, p_\nu)}{\partial p_\mu} \dot{p}_\mu \right) + \frac{\partial f(q_\nu, p_\nu)}{\partial \beta} \\
&= \sum_{\mu} \left( \frac{\partial f(q_\nu, p_\nu)}{\partial q_\mu} \frac{\partial K(q_\nu, p_\nu)}{\partial p_\mu} - \frac{\partial f(q_\nu, p_\nu)}{\partial p_\mu} \frac{\partial K(q_\nu, p_\nu)}{\partial q_\mu} \right) + \frac{\partial f(q_\nu, p_\nu)}{\partial \beta}. \tag{2.21}
\end{align}

Since \( f \) usually does not explicitly depend on \( \beta \), in most situations the last term \( \frac{\partial f(q_\nu, p_\nu)}{\partial \beta} \) would vanish, and we are left with
\begin{align}
\dot{f}(q_\nu, p_\nu) &= \sum_{\mu} \left( \frac{\partial f(q_\nu, p_\nu)}{\partial q_\mu} \frac{\partial K(q_\nu, p_\nu)}{\partial p_\mu} - \frac{\partial f(q_\nu, p_\nu)}{\partial p_\mu} \frac{\partial K(q_\nu, p_\nu)}{\partial q_\mu} \right). \tag{2.22}
\end{align}

We define this to be the extended Poisson brackets of \( f(q_\nu, p_\nu) \) and \( K(q_\nu, p_\nu) \), denoted as \( \{ f, K \} \) and we have
\begin{align}
\dot{f} &= \{ f, K \}. \tag{2.23}
\end{align}
This implies that the quantity $f$ is a conserved quantity or constant of motion with $\dot{f} = 0$ when it has a vanishing Poisson bracket with the extended Hamiltonian function $\mathcal{K}$.

In general, the extended Poisson bracket of two phase-space functions $g(q_\nu, p_\nu)$ and $h(q_\nu, p_\nu)$ is

$$\{g, h\} = \sum_{\mu=0}^{D} \left( \frac{\partial g(q_\nu, p_\nu)}{\partial q_\mu} \frac{\partial h(q_\nu, p_\nu)}{\partial p_\mu} - \frac{\partial g(q_\nu, p_\nu)}{\partial p_\mu} \frac{\partial h(q_\nu, p_\nu)}{\partial q_\mu} \right).$$

(2.24)

The extended Poisson brackets between $q$’s, $q$ and $p$, and $p$’s are

$$\{q_\mu, q_\nu\} = 0 \quad \{q_\mu, p_\nu\} = \delta_{ij} \quad \{p_\mu, p_\nu\} = 0.$$  

(2.25)
3 Spacetime Representation of Quantum Mechanics

3.1 Formalism in Quantum Mechanics

Based on the extended theories outlined above, we are now ready to develop a quantum formalism that treats time as a dynamical variable.

3.1.1 Spacetime Representation

In this subsection, we develop a spacetime representation that treats space and time equally. First we promote Poisson brackets to commutators of quantum operators. In general, commutators of quantum operators have an algebraic structure that closely resembles the fundamental Poisson brackets of the classical variables. Poisson brackets can be promoted to quantum commutators through the procedure of canonical quantization [7],

\[
\{ f, g \} \rightarrow \frac{[\hat{f}, \hat{g}]}{i\hbar},
\]

where the hat notation represents operators. This allows us to transit the extended Poisson brackets in Eq. (2.25) to the following canonical commutation relations,

\[
[\hat{q}_\mu, \hat{q}_\nu] = 0, \quad [\hat{q}_\mu, \hat{p}_\nu] = i\hbar\delta_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0.
\]  

(3.1)

In spacetime, \( \mu, \nu = 0, 1, 2, 3 \).

The new spacetime representation under \( q, p \) coordinates can then be developed as follows. Without loss of generality, we will focus on time and one spatial dimension. Consider the spacetime coordinates of \( q_0 \) and \( q_1 \), and let \( p_0 \) and \( p_1 \) be the corresponding momentum coordinates. In the Schrödinger picture, position operators are equivalent to the classical variables
in the classical mechanics. Hence

\[ \langle q_0, q_1 | \hat{q}_0 | \psi \rangle = q_0 \langle q_0, q_1 | \psi \rangle, \quad (3.2) \]

\[ \langle q_0, q_1 | \hat{q}_1 | \psi \rangle = q_1 \langle q_0, q_1 | \psi \rangle, \quad (3.3) \]

where \( \psi \) represents a generic state. This is because the commutation relation \([\hat{q}_i, \hat{q}_j] = 0\) implies that the eigenfunctions of either operator can be chosen to be simultaneous eigenfunctions of the other; thus the eigenstates can be chosen to be simultaneous eigenstates of both \( \hat{q}_0 \) and \( \hat{q}_1 \). The state \( |q_0, q_1\rangle \) is different from the conventional approach in quantum mechanics as \( |q_0, q_1\rangle \) refers to an eigenstate of both \( \hat{q}_0 \) and \( \hat{q}_1 \), whereas in conventional quantum mechanics the states analogous to \( |q_0, q_1\rangle \), which is \( |x(t)\rangle \), refers to the eigenstate of \( |x\rangle \) at the time \( t \).

### 3.1.2 Generator for Coordinate Translation

We can now find the operator for space and time translations from the commutation relations derived in the previous subsection.

Consider the phase-space function \( f = e^{\alpha \hat{p}_i} \hat{q}_i e^{-\alpha \hat{p}_i} \) where \( \alpha \equiv -\frac{iq_i}{\hbar} \). Applying the Baker-Campbell-Hausdorff formula [8] on it, we get

\[ e^{\alpha \hat{p}_i} \hat{q}_i e^{-\alpha \hat{p}_i} = \hat{q}_i + [\hat{q}_i, \alpha \hat{p}_i] + 1/2! [\hat{q}_i, [\hat{q}_i, \alpha \hat{p}_i]] + 1/3! [\hat{q}_i, [\hat{q}_i, [\hat{q}_i, \alpha \hat{p}_i]]] + \ldots \]

But from Eq. (3.1), \([\hat{q}_i, \hat{p}_i] = i\hbar\). An algebraic deduction would show that \([\hat{q}_i, [\hat{p}_i, \hat{p}_i]]\), and likewise all the other terms, reduce to zero. The equation then becomes \( e^{\alpha \hat{p}_i} \hat{q}_i e^{-\alpha \hat{p}_i} = \hat{q}_i + i\hbar \alpha \). Multiplying \( e^{-\alpha \hat{p}_i} \) from the left, we get

\[ \hat{q}_i e^{-\alpha \hat{p}_i} = e^{-\alpha \hat{p}_i} \hat{q}_i + i\hbar \alpha e^{-\alpha \hat{p}_i}. \]
If we apply this on a ket $|q_i\rangle$ where $\hat{q}_i|q_i\rangle = q_i|q_i\rangle$, we get

$$\hat{q}_i e^{-\alpha \hat{p}_i} |q_i\rangle = (q_i + i\hbar \alpha) e^{-\alpha \hat{p}_i} |q_i\rangle.$$ 

Using the definition of $\alpha$,

$$\hat{q}_i e^{\frac{iq'_i}{\hbar}} |q_i\rangle = (q_i + q'_i) e^{\frac{iq'_i}{\hbar}} |q_i\rangle.$$ 

Therefore, $(q_i + q'_i)$ are the eigenvalues for the eigenstates $e^{\frac{iq'_i}{\hbar}} |q_i\rangle$ under $\hat{p}_i$. When the operator $\hat{q}_i$ acts on the state $|q_i + q'_i\rangle$, we get

$$\hat{q}_i |q_i + q'_i\rangle = (q_i + q'_i) |q_i + q'_i\rangle,$$ 

and

$$e^{\frac{iq'_i}{\hbar}} |q_i\rangle = |q_i + q'_i\rangle.$$ 

Therefore, the translation operator in the $q$ coordinate is $e^{iq'_i \hat{p}_i / \hbar}$. By symmetry, the operator for the $p$ coordinate is $e^{ip'_i \hat{q}_i / \hbar}$. For $q'_0$, we get a operator of time translation, $e^{ip_0 \hat{q}_0 / \hbar}$.

### 3.1.3 Momentum in Spacetime Representation

In this section we find $\hat{p}_0, \hat{p}_1$ in the $q_0, q_1$ representation. First, let us define $|0_q\rangle$ to be the simultaneous eigenkets of $\hat{q}_0$ and $\hat{q}_1$ with eigenvalues $q_i = 0$, and $|0_p\rangle$ the eigenkets of $\hat{p}_0$ and $\hat{p}_1$ with eigenvalues $p_i = 0$. Applying the translational generator, we get

$$\langle q'_0, q'_1 | p'_0, p'_1 \rangle = \langle 0_q | e^{ip_0 \hat{q}_0 / \hbar} e^{ip_1 \hat{q}_1 / \hbar} | p'_0, p'_1 \rangle$$

$$= e^{ip'_0 \hat{q}_0 / \hbar} e^{ip'_1 \hat{q}_1 / \hbar} \langle 0_q | p'_0, p'_1 \rangle.$$ 

As $\hat{p}_0, \hat{p}_1$ share the same eigenstates, the ket $|p'_0, p'_1\rangle$ can be written as $e^{ip_0 \hat{q}_0 / \hbar} e^{ip_1 \hat{q}_1 / \hbar} |0_p\rangle$. Then
the right hand side of Eq. (3.6) becomes

\[ e^{ip_0 \frac{q}{\hbar}} e^{ip_1 q_1 \frac{q}{\hbar}} \langle 0_q | p_0', p_1' \rangle = e^{ip_0 q_0 / \hbar} e^{ip_1 q_1 / \hbar} \langle 0_q | 0_p \rangle. \]

We can apply \( e^{-i p_0 q_0 / \hbar} e^{-i p_1 q_1 / \hbar} \) on \( \langle 0_q | \) to get \( e^0 = 1 \), and we are left with

\[ \langle q_0', q_1' | p_0', p_1' \rangle = e^{ip_0 q_0 / \hbar} e^{ip_1 q_1 / \hbar} \langle 0_q | 0_p \rangle, \]  

(3.7)

where \( \langle 0_q | 0_p \rangle \) can be determined through

\[ \langle p_0', p_1' | p_0''', p_1''' \rangle = \delta(p_0' - p_0'') \delta(p_1' - p_1'''). \]  

(3.8)

By completeness of \( |q_0, q_1'\rangle \), \( \int \int dq_0 dq_1' | q_0, q_1' \rangle \langle q_0, q_1' | = 1 \), and we have

\[ \langle p_0', p_1' | p_0''', p_1''' \rangle = \int \int dq_0 dq_1' \langle p_0', p_1' | q_0, q_1' \rangle \langle q_0', q_1' | p_0'', p_1''' \rangle. \]  

(3.9)

On the other hand, by definition of delta function, \( \delta(q' - q'') = \int dk \frac{1}{2\pi} e^{ik(q' - q'')} \) where \( k \) can be any variable [9]. Then we have

\[ \delta(p_0' - p_0'') = \frac{1}{2\pi \hbar} \int dq_0' e^{i(p_0' - p_0'') q_0' / \hbar}, \]

\[ \delta(p_1' - p_1'') = \frac{1}{2\pi \hbar} \int dq_1' e^{i(p_1' - p_1'') q_1' / \hbar}, \]

and Eq. (3.9) becomes

\[ \int \int dq_0' dq_1' \langle p_0', p_1' | q_0', q_1' \rangle \langle q_0', q_1' | p_0'', p_1''' \rangle = \frac{1}{(2\pi \hbar)^2} \int \int dq_0' dq_1' e^{i(p_0' q_0' + p_1' q_1') / \hbar} e^{-i(p_0'' q_0' + p_1'' q_1') / \hbar}, \]  

(3.10)

from which we get

\[ \langle q_0', q_1' | p_0', p_1' \rangle = \frac{1}{2\pi \hbar} e^{-i(p_0' q_0' + p_1' q_1') / \hbar}. \]  

(3.11)
Now we consider an arbitrary ket $|a'|$ of a one-particle system. The probability amplitude of finding the system at the point $q'_0$ and $q'_1$ in spacetime is given by $\langle q'_0, q'_1 | a' \rangle$, and the probability amplitude of finding the system at momenta $p'_0$ and $p'_1$ is $\langle p'_0, p'_1 | a' \rangle$. The state $|a'\rangle$ can be written in the coordinate basis as

$$
\langle q'_0, q'_1 | a' \rangle = \int \int dp'_0 dp'_1 \langle q'_0, q'_1 | p'_0, p'_1 \rangle \langle p'_0, p'_1 | a' \rangle.
$$

Using Eq. (3.11), we get

$$
\langle q'_0, q'_1 | a' \rangle = \int \int dp'_0 dp'_1 \frac{1}{2\pi\hbar} e^{-i(p'_0 q'_0 + p'_1 q'_1)/\hbar} \langle p'_0, p'_1 | a' \rangle.
$$

We can translate this into wave mechanics as

$$
\psi_a(q'_0, q'_1) = \frac{1}{2\pi\hbar} \int \int dp'_0 dp'_1 e^{-i(p'_0 q'_0 + p'_1 q'_1)/\hbar} \phi_{a'}(p'_0, p'_1), \quad (3.12)
$$

where $\psi_a(q'_0, q'_1) = \langle q'_0, q'_1 | a' \rangle$ is the wave function in coordinate basis and $\phi_{a'}(p'_0, p'_1) = \langle p'_0, p'_1 | a' \rangle$ is the state $|a'\rangle$ in the momentum basis. Now consider the action of the momentum operators $\hat{p}_0$ and $\hat{p}_1$ on $\psi$:

$$
\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | a' \rangle = \int \int dq''_0 dq''_1 \langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle \langle q''_0, q''_1 | a' \rangle. \quad (3.13)
$$

We now consider $\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle$. By completeness of the state $|p'_0, p'_1\rangle$,

$$
\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle = \int \int dp'_0 dp'_1 \langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | p'_0, p'_1 \rangle \langle p'_0, p'_1 | q''_0, q''_1 \rangle.
$$

Applying $\hat{p}_0 + \hat{p}_1$ on the state $|p'_0, p'_1\rangle$, the right hand side becomes

$$
\int \int dp'_0 dp'_1 \langle q'_0, q'_1 | p'_0, p'_1 \rangle (p'_0 + p'_1) \langle p'_0, p'_1 | q''_0, q''_1 \rangle.
$$
Using (3.10), it then becomes

\[
\frac{1}{(2\pi\hbar)^2} \int \int dp'_0 \, dp'_1 \, (p'_0 + p'_1) \, e^{-i(p'_0q'_0 + p'_1q'_1 - p_0q_0 - p_1q_1)/\hbar} = \frac{1}{(2\pi\hbar)^2} \int \int dp'_0 \, dp'_1 \, (p'_0 + p'_1) \, e^{i(p'_0(q'_0 - q_0) + p'_1(q'_1 - q_1))/\hbar}.
\]

Therefore,

\[
\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle = \frac{1}{(2\pi\hbar)^2} \int \int dp'_0 \, dp'_1 \, (p'_0 + p'_1) \, e^{i(p'_0(q'_0 - q_0) + p'_1(q'_1 - q_1))/\hbar},
\]

where we have used the notation \( \mathbf{p} = (p_0, p_1) \). Apply \( \delta(q' - q'') = \int \frac{dk}{2\pi} e^{ik(q' - q'')} \) again, and let \( p'_0 = \hbar k'_0 \) so that \( dp'_0 = \hbar dk'_0 \), we can rewrite Eq. (3.14) as

\[
\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle = \frac{1}{(2\pi\hbar)^2} \int \int dp'_0 \, dp'_1 \, (p'_0 + p'_1) \, e^{i(p'_0(q'_0 - q_0) + p'_1(q'_1 - q_1))/\hbar} = \frac{1}{(2\pi\hbar)^2} \int \int \hbar^2 \, dk'_0 \, dk'_1 \, (k'_0 + k'_1) \, e^{ik'_0(q'_0 - q_0)} \, e^{ik'_1(q'_1 - q_1)}.
\]

Separating the integrals, we get

\[
\frac{\hbar}{(2\pi)^2} \left[ \int dk'_0 \, k'_0 \, e^{ik'_0(q'_0 - q_0)} \int dk'_1 \, e^{ik'_1(q'_1 - q_1)} + \int dk'_0 \, e^{ik'_0(q'_0 - q_0)} \int dk'_1 \, e^{ik'_1(q'_1 - q_1)} + \int dk'_0 \, e^{ik'_0(q'_0 - q_0)} \right].
\]

Pull \( k'_0 \) and \( k'_1 \) out of the integral,

\[
\frac{\hbar}{(2\pi)^2} \left[ \frac{1}{i} \frac{\partial}{\partial q'_0} \int dk'_0 \, e^{ik'_0(q'_0 - q_0)} \int dk'_1 \, e^{ik'_1(q'_1 - q_1)} + \int dk'_0 \, e^{ik'_0(q'_0 - q_0)} \frac{1}{i} \frac{\partial}{\partial q'_1} \int dk'_1 \, e^{ik'_1(q'_1 - q_1)} \right].
\]

Using the definition of the delta function again, we can write

\[
\langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | q''_0, q''_1 \rangle = \frac{\hbar}{i} \left( \frac{\partial}{\partial q'_0} \delta(q'_0 - q'_0) \delta(q'_1 - q''_1) + \delta(q'_0 - q'_0) \frac{\partial}{\partial q'_1} \delta(q'_1 - q''_1) \right).
\]
Substituting this into the right hand side of Eq. (3.13), it becomes

\[ \frac{\hbar}{i} \int \int dq''_0 dq''_1 \left( \frac{\partial}{\partial q''_0} \delta(q''_0 - q'_0) \delta(q''_1 - q'_1) + \delta(q''_0 - q'_0) \frac{\partial}{\partial q''_1} \delta(q''_1 - q'_1) \right) \langle q''_0, q''_1 | a' \rangle. \]

Breaking the integrals into two parts, we get

\[ \frac{\hbar}{i} \left[ \frac{\partial}{\partial q''_0} \int \int dq''_0 dq''_1 \delta(q''_0 - q'_0) \delta(q''_1 - q'_1) \langle q''_0, q''_1 | a' \rangle \right] \]

and

\[ \frac{\hbar}{i} \left[ \frac{\partial}{\partial q''_1} \int \int dq''_0 dq''_1 \delta(q''_0 - q'_0) \delta(q''_1 - q'_1) \langle q''_0, q''_1 | a' \rangle \right]. \]

Therefore when \( q'_0 = q''_0 \) and \( q'_1 = q''_1 \),

\[ \langle q'_0, q'_1 | \hat{p}_0 + \hat{p}_1 | a' \rangle = \frac{\hbar}{i} \left( \frac{\partial}{\partial q'_0} \langle q'_0, q'_1 | a' \rangle + \frac{\partial}{\partial q'_1} \langle q'_0, q'_1 | a' \rangle \right). \]

(3.15)

So finally,

\[ \langle q_0, q_1 | \hat{p}_0 | \psi \rangle = -i \hbar \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle, \]

(3.16)

\[ \langle q_0, q_1 | \hat{p}_1 | \psi \rangle = -i \hbar \frac{\partial}{\partial q_1} \langle q_0, q_1 | \psi \rangle. \]

(3.17)

In general, we could apply the commutation relation \([\hat{q}_i, \hat{p}_i] = i \hbar\) on any state in the \( q \)-representation such that \( \psi = \langle q | \psi \rangle \)

\[ \langle q | [\hat{q}_i, \hat{p}_i] | \psi \rangle = [\hat{q}_i, \hat{p}_i] \psi = \frac{\hbar}{i} \left( q_i \frac{\partial \psi}{\partial q_i} - \frac{\partial}{\partial q_i} \psi \right) = i \hbar \psi. \]

A similar analysis for \( \hat{p} \) gives the form

\[ \langle p | [\hat{q}_i, \hat{p}_i] | \phi \rangle = [\hat{q}_i, \hat{p}_i] \phi = i \hbar \left( \frac{\partial \phi}{\partial p_i} p_i \phi - p_i \frac{\partial \phi}{\partial p_i} \right) = i \hbar \phi, \]

where \( \phi \) in the \( p \)-representation such that \( \phi = \langle p | \psi \rangle \). Thus, we conclude that for analytical
functions $F(\hat{p})$ and $G(\hat{q})$ where $F = a_0 + a_1\hat{p} + a_2\hat{p}^2...$ and $G = b_0 + b_1\hat{q} + b_2\hat{q}^2...$, it holds that

$$[\hat{q}_i, F(\hat{p})] = i\hbar \frac{\partial F}{\partial \hat{p}_i},$$

and

$$[\hat{p}_i, G(\hat{q})] = \frac{\hbar}{i} \frac{\partial G}{\partial \hat{q}_i}. \quad (3.18)$$

### 3.2 Deriving the Schrödinger Equation

With the new formalism, we can now study the spacetime behavior under the translations of other operators. We start from the extended Hamiltonian, $\hat{K}$, and let it act on an state $|\psi\rangle$. For physical states, $\hat{K} = \hat{p}_0 + \hat{H}$. Therefore,

$$\langle q_0, q_1 | \hat{K} | \psi \rangle = \langle q_0, q_1 | \hat{p}_0 + \hat{H} | \psi \rangle$$

By Eq. (3.16),

$$\langle q_0, q_1 | \hat{p}_0 + \hat{H} | \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle + \hat{H} \langle q_0, q_1 | \psi \rangle,$$

However, because $\hat{K} | \psi \rangle = 0$, therefore

$$\frac{\hbar}{i} \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle + \hat{H} \langle q_0, q_1 | \psi \rangle = 0,$$

or

$$i\hbar \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle = \hat{H} \langle q_0, q_1 | \psi \rangle.$$

This is the Schrödinger equation. For example, if $\hat{H} = \frac{\hat{p}_1^2}{2m} + V(\hat{q}_1)$, then we get

$$i\hbar \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial q_1} \right)^2 \langle q_0, q_1 | \psi \rangle + V(q_1) \langle q_0, q_1 | \psi \rangle,$$

and we can write this in wave mechanics as

$$i\hbar \frac{\partial}{\partial q_0} \psi(q_0, q_1) = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial q_1} \right)^2 \psi(q_0, q_1) + V(q_1) \psi(q_0, q_1).$$
3.3 Propagator

Analogous to the propagator in the traditional quantum mechanics, the propagator in our formalism can be written as

\[ \langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle. \]

When \( \beta = \beta' \), the inner product of \( |q_0, q_1 \rangle \) with \( |q'_0, q'_1 \rangle \) is zero unless \( q_0 = q'_0 \) and \( q_1 = q'_1 \). Thus the propagator for \( \beta = \beta' \) is

\[ \langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \delta(q_0 - q'_0)\delta(q_1 - q'_1). \]

To find the propagator for \( \beta \neq \beta' \), we begin by defining a beta-evolution operator \( \hat{U} \), in order to find out how any arbitrary state ket would change as \( \beta \) evolves. Note that \( \hat{U} \) is an unitary operator, which satisfies the condition \( \hat{U}^\dagger(\beta, \beta_0) \hat{U}(\beta, \beta_0) = 1 \). This ensures that if the initial state is normalized, all states at later betas are also normalized. Moreover, the compositional rule of unitary operators is [5]

\[ \hat{U}(\beta_2, \beta_0) = \hat{U}(\beta_2, \beta_1) \hat{U}(\beta_1, \beta_0). \]

We now look at the evolution of \( \beta \) under \( \hat{U}(\beta, \beta_0) \). For convenience, we set \( \beta_0 \) to be zero, so that \( \hat{U}(\beta, \beta_0) \) becomes \( \hat{U}(\beta) \). We derive the fundamental equations of motion in the Heisenberg Picture, and promote the Poisson brackets of Eq. (2.22) to a commutator. We obtain

\[ \frac{d \hat{A}^H}{d\beta} = \frac{1}{i\hbar} [\hat{A}^H, \hat{K}^H], \]

where \( A \) is an observable and superscript \( H \) stands for Heisenberg Picture. Unlike the Schrödinger Picture, in which operators act on quantum states and shift them, the Heisenberg Picture treats states as unchanged and let operators and their corresponding observables vary with \( \beta \). This gives the Heisenberg observable \( A^H \) as

\[ \hat{A}^H(\beta) \equiv \hat{U}(\beta)\hat{A}^S \hat{U}^\dagger(\beta), \]
where superscript \( S \) represents the Schrödinger Picture. Using the above definitions, we can expand the right side of Eq. (3.19) in the following way,

\[
\frac{1}{i\hbar} [\hat{A}^H, \hat{K}^H] = \frac{1}{i\hbar} [\hat{A}^H, \hat{U}^\dagger \hat{K} \hat{U}]
\]

\[
= \frac{1}{i\hbar} \hat{A}^H \hat{U}^\dagger \hat{K} \hat{U} - \frac{1}{i\hbar} \hat{U}^\dagger \hat{K} \uvec{\hat{A}}^H
\]

\[
= \frac{1}{i\hbar} \hat{U}^\dagger \hat{A}^S \hat{K} \hat{U} - \frac{1}{i\hbar} \hat{U}^\dagger \hat{K} \hat{A}^s \hat{U}.
\]

(3.21)

For clarity, we have omitted the hat on \( \uvec{\hat{U}} \); \( \hat{U} \) now represents the unitary operator \( \uvec{\hat{U}} \). If we take the derivative of Eq. (3.20) with respect to \( \beta \), the left hand side of Eq. (3.19) becomes

\[
\frac{d\hat{A}^H}{d\beta} = \frac{\partial \hat{U}^\dagger}{\partial \beta} \hat{A}^S \hat{U} + \hat{U}^\dagger \frac{\partial \hat{A}^S}{\partial \beta} \frac{d\hat{U}}{d\beta} + \hat{U}^\dagger \frac{\partial \hat{A}^S}{\partial \beta} \hat{U}.
\]

(3.22)

Since \( \hat{A} \) does not explicitly depend on \( \beta \), the last term vanishes. Combining Eq. (3.21) and Eq. (3.22), the arrangement that would yield the simplest solution is

\[
\frac{\partial \hat{U}}{\partial \beta} = \frac{1}{i\hbar} \hat{K} \hat{U} \quad \text{and} \quad \frac{\partial \hat{U}^\dagger}{\partial \beta} = -\frac{1}{i\hbar} \hat{U}^\dagger \hat{K}
\]

(3.23)

Notice that the second equation in Eq. (3.23) is just the complex conjugate of the first. Rearrange the first equation, we get

\[
i\hbar \frac{\partial}{\partial \beta} \hat{U} = \hat{K} \hat{U}.
\]

(3.24)

which is just the Schrödinger equation for \( \hat{U} \).

We now show the effect of \( \hat{U} \) on an arbitrary state \( |\alpha\rangle \). We write Eq. (3.24) in a more explicit form

\[
i\hbar \frac{\partial}{\partial \beta} \hat{U}(\beta, \beta_0) = \hat{K} \hat{U}(\beta, \beta_0)
\]

(3.25)
and solve for $U$,

$$dU(\beta, \beta_0) = -\frac{i\hat{K}}{\hbar} U(\beta, \beta_0) \, d\beta,$$

$$U(\beta + d\beta, \beta_0) - U(\beta, \beta_0) = -\frac{i\hat{K}}{\hbar} U(\beta, \beta_0) \, d\beta,$$

$$U(\beta + d\beta, \beta_0) = (1 - \frac{i\hat{K}}{\hbar} d\beta) U(\beta, \beta_0). \quad (3.26)$$

Applying the compositional property of unitary operators, the left side of Eq. (3.26) becomes

$$U(\beta + d\beta, \beta_0) = U(\beta + d\beta, \beta) \cdot U(\beta, \beta_0).$$

Substituting it back into Eq.(3.26), we get

$$U(\beta + d\beta, \beta) = 1 - \frac{i\hat{K}}{\hbar} d\beta.$$

Taking limits on both sides, we have

$$\lim_{d\beta \to 0} U(\beta + d\beta, \beta) = 1. \quad (3.27)$$

Now consider a state $|\alpha\rangle$ at $\beta_0$, or $|\alpha\rangle = |\alpha, \beta_0\rangle$. Applying Eq. (3.27) on the state, we get

$$\lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) \, |\alpha\rangle = \lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) \, |\alpha, \beta_0\rangle = |\alpha, \beta_0\rangle,$$

where we used Eq. (3.27) in the last step. We denote $|\alpha, \beta_0; \beta_0 + d\beta\rangle$ to be the state after the movement that the initial state $|\alpha, \beta_0\rangle$ shifts from $\beta_0$ to $\beta + d\beta$, and we have

$$\lim_{d\beta \to 0} |\alpha, \beta_0; \beta_0 + d\beta\rangle = |\alpha, \beta_0\rangle.$$

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Since \( \lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) = 1 \), it is safe to multiply the right side by \( \lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) \)

\[
\lim_{d\beta \to 0} |\alpha, \beta_0; \beta_0 + d\beta \rangle = \lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) |\alpha, \beta_0 \rangle.
\]

which essentially reduces to

\[
|\alpha, \beta_0; \beta_0 + d\beta \rangle = U(\beta_0 + d\beta, \beta_0) |\alpha, \beta_0 \rangle. \tag{3.28}
\]

Eq. (3.28) tells how \( U \) acts on the initial state \( |\alpha, \beta_0 \rangle \). Moreover, it shows that the state remains unchanged under the translation of operators, which is indeed how Heisenberg Picture interprets the quantum dynamics.

We will now derive formal solutions to the Schrödinger equation Eq. (3.25) for the beta-evolution operator. We consider three separate cases [5].

Case 1. The \( \hat{K} \) operator has no \( \beta \) dependence. This is the simplest case, and we could solve the Schrödinger equation by successively compounding \( N \) infinitesimal beta-evolution operators in Eq. (3.26), each of which is characterized by the beta interval \( \frac{\beta - \beta_0}{N} \):

\[
\lim_{N \to \infty} [1 - \frac{(i\hat{K}) (\beta - \beta_0)}{N}]^N = e^{-i\hat{K}(\beta - \beta_0)}.
\]

Case 2. The Hamiltonian operator \( \hat{K} \) has \( \beta \) dependence, and the \( \hat{K} \)'s of different \( \beta \) values commute. The solution in this case is [5]

\[
U(\beta_0 + d\beta, \beta_0) = e^{-\frac{i}{\hbar} \int_{\beta_0}^{\beta} d\beta' \hat{K}(\beta')} \tag{3.29}
\]

To prove this, first we Taylor expand the exponential of the right side,

\[
e^{-\frac{i}{\hbar} \int_{\beta_0}^{\beta} d\beta' \hat{K}(\beta')} = 1 + \frac{-i}{\hbar} \int_{\beta_0}^{\beta} d\beta' \hat{K}(\beta') + \left[ \frac{(-i)^2}{2} \right] \left[ \frac{1}{\hbar} \int_{\beta_0}^{\beta} d\beta' \hat{K}(\beta') \right]^2 + \cdots.
\]
The derivative of this expansion with respect to $\beta$ is
\[
\frac{\partial}{\partial \beta} e^{-\frac{i}{\hbar} \int_{\beta_0}^{\beta} d\beta' \hat{K}(\beta')} = -i \frac{\hat{K}}{\hbar} + \left[ \frac{(-i)^2}{2} \right] 2 \left( \frac{\hat{K}^2}{\hbar} \right) (\beta - \beta_0) + \cdots,
\]
Therefore Eq. (3.29) satisfies Schrödinger’s equation.

Case 3. The Hamiltonian operator $\hat{K}$ has $\beta$ dependence, but the $\hat{K}$’s at different betas do not commute. In this case the solution would be the Dyson series [5]
\[
\mathcal{U}(\beta_0 + d\beta, \beta_0) = 1 + \sum_{n=1}^{\infty} \left[ -\frac{i}{\hbar} \right]^n \int_{\beta_0}^{\beta} d\beta_1 \int_{\beta_0}^{\beta_1} d\beta_2 \cdots \int_{\beta_0}^{\beta_{n-1}} d\beta_n \hat{K}(\beta_1) \hat{K}(\beta_2) \cdots \hat{K}(\beta_n). \tag{3.30}
\]
For a detailed proof, see [5, 168].

Now we return to the question of the propagator $\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle$ when $\beta \neq \beta'$. We assume that the eigenstates of $\hat{K}$ are complete so that $\sum_k |n_k\rangle\langle n_k| = 1$. By summing over the eigenstates of $\hat{K}$, we can write $\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle$ as $\sum_{n_k} \langle q_0, q_1, \beta | n_k \rangle \langle n_k | q'_0, q'_1, \beta' \rangle$. Note that for each individual state $n_k$, $|n\rangle = |n_{q_0}\rangle|n_{q_1}\rangle$ in order to form a complete basis. Using the beta evolution operator, we have
\[
\sum_{n_k} \langle q_0, q_1, \beta | n_k \rangle \langle n_k | q'_0, q'_1, \beta' \rangle = \sum_{n_k} \langle q_0, q_1, 0 | e^{-\frac{i \hat{K}}{\hbar}} | n_k \rangle \langle n_k | e^{\frac{i \hat{K}}{\hbar}} q'_0, q'_1, 0 \rangle.
\]
The states $|n_k\rangle$ are defined such that $\hat{K}|n_k\rangle = K_{n_k} |n_k\rangle$. Then we have
\[
\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \sum_{n_k} e^{-\frac{i K_{n_k} (\beta - \beta')}{\hbar}} \langle q_0, q_1, 0 | n_k \rangle \langle n_k | q'_0, q'_1, 0 \rangle.
\]
This result is the propagator in the new formalism. It can be further simplified by noting that $\hat{K}|n_k\rangle = (\hat{p}_0 + \hat{H})|n_k\rangle = 0$, and therefore all the eigenvalues $K_{n_k}$ are zero. Then,
\[
\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \sum_{n_k} \langle q_0, q_1, 0 | n_k \rangle \langle n_k | q'_0, q'_1, 0 \rangle.
\]
Because \( \langle q_0, q_1 | n_k \rangle = \psi_n(q_1)e^{\frac{i}{\hbar}E_nq_0} \), we have

\[
\langle q_0, q_1, \beta | q_0', q_1', \beta' \rangle = \sum_n \psi_n(q_1)e^{\frac{i}{\hbar}E_nq_0}\psi^*_n(q_1)e^{-\frac{i}{\hbar}E_nq_0}\\= \sum_n \psi^*_n(q_1)\psi_n(q_1)e^{\frac{i}{\hbar}E_n(q_0-q_0')},
\]

which is the standard form for propagators.

### 3.4 Uncertainty Principle

Another fundamental question in quantum mechanics is the Uncertainty Principle. We now derive the \((q_0, p_0)\) uncertainty relation. We will show the standard method for deriving uncertainty relations, then apply it to the \((q_0, p_0)\) case.

We begin with the Schwarz Inequality [5], \( \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2 \). Let \( A \) and \( B \) be two observables such that \( \Delta \hat{A} | \psi \rangle = |\alpha \rangle \) and \( \Delta \hat{B} | \psi \rangle = |\beta \rangle \), where \( \Delta \hat{A} \equiv A - \langle A \rangle \) and \( \Delta \hat{B} = B - \langle B \rangle \). Because \( \Delta \hat{A} \) and \( \Delta \hat{B} \) are Hermitian, the Schwarz Inequality can be written as

\[
\langle \psi | \left( \Delta \hat{A} \right)^2 | \psi \rangle \langle \psi | \left( \Delta \hat{B} \right)^2 | \psi \rangle \geq |\langle \psi | \Delta \hat{A} \Delta \hat{B} | \psi \rangle|^2. \tag{3.31}
\]

On the right hand side, \( \Delta \hat{A} \Delta \hat{B} \) can be written as a linear combination

\[
\Delta \hat{A} \Delta \hat{B} = \frac{1}{2} [\Delta \hat{A}, \Delta \hat{B}] + \frac{1}{2} \{ \Delta \hat{A}, \Delta \hat{B} \}, \tag{3.32}
\]

where \( \{ \Delta \hat{A}, \Delta \hat{B} \} \) is the anticommutator. We can verify this equation by expanding the right hand side of it,

\[
\frac{1}{2}(\Delta \hat{A} \Delta \hat{B} - \Delta \hat{B} \Delta \hat{A}) + \frac{1}{2}(\Delta \hat{A} \Delta \hat{B} + \Delta \hat{B} \Delta \hat{A}),
\]

which equals to \( \Delta \hat{A} \Delta \hat{B} \) after cancellation.
In Eq. (3.32), the commutator $[\Delta \hat{A}, \Delta \hat{B}]$ is anti-Hermitian, as

$$([\Delta \hat{A}, \Delta \hat{B}])^\dagger = (\Delta \hat{A}\Delta \hat{B} - \Delta \hat{B}\Delta \hat{A})^\dagger = \Delta \hat{B}\Delta \hat{A} - \Delta \hat{A}\Delta \hat{B} = -[\Delta \hat{A}, \Delta \hat{B}].$$

On the other hand, the anticommutator $\{\Delta \hat{A}, \Delta \hat{B}\}$ is Hermitian. Thus,

$$\langle \psi | \Delta \hat{A}\Delta \hat{B} | \psi \rangle = \frac{1}{2} \langle \psi | [\Delta \hat{A}, \Delta \hat{B}] | \psi \rangle + \frac{1}{2} \langle \psi | \{\Delta \hat{A}, \Delta \hat{B}\} | \psi \rangle.$$

The first part on the right hand side of this equation is purely imaginary because the expectation value of an anti-Hermitian operator is imaginary, whereas the second part is purely real because the expectation value of a Hermitian operator is real. Squaring the whole equation, the real part becomes

$$\left| \left( \langle \psi | \Delta \hat{A}\Delta \hat{B} | \psi \rangle \right) \right|^2 = \frac{1}{4} \left| \langle \psi | [\Delta \hat{A}, \Delta \hat{B}] | \psi \rangle \right|^2 + \frac{1}{4} \left| \langle \psi | \{\Delta \hat{A}, \Delta \hat{B}\} | \psi \rangle \right|^2.$$

The left hand side of Eq. (3.31) is greater than the left hand side of the above equation; then it must also be greater than either one of the terms on the right hand side of the above equation, so we have

$$\langle \psi | (\Delta \hat{A})^2 | \psi \rangle \langle \psi | (\Delta \hat{B})^2 | \psi \rangle \geq \frac{1}{4} \left| \langle \psi | [\Delta \hat{A}, \Delta \hat{B}] | \psi \rangle \right|^2.$$

We can simply substitute $[\Delta \hat{A}, \Delta \hat{B}]$ by $[\hat{A}, \hat{B}]$, and obtain the uncertainty relation.

Now we apply this standard method to the case of $q_0$ and $p_0$, of which the commutator relation is $[q_0, p_0] = i\hbar$. Thus,

$$\langle \psi | (\Delta \hat{q}_0)^2 | \psi \rangle \langle \psi | (\Delta \hat{p}_0)^2 | \psi \rangle \geq \frac{1}{4} \left| \langle i\hbar \rangle \right|^2 = \frac{1}{4} |i\hbar|^2,$$

or

$$\langle \psi | \Delta \hat{q}_0 | \psi \rangle \langle \psi | \Delta \hat{p}_0 | \psi \rangle \geq \frac{\hbar}{2}. \tag{3.34}$$
For the generic case, the position and momentum observables with the commutation relation 
\[ [q_0, p_0] = i\hbar \] have continuous eigenvalues for any state \(|\psi\rangle\), and the probability amplitudes 
\[ \langle q_0|\psi\rangle \] and \[ \langle p_0|\psi\rangle \] are Fourier transforms of each other

\[
\langle q_0|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int \langle p_0|\psi\rangle e^{(ip_0q_0)/\hbar} dp_0.
\]

It follows that the widths of the probability densities \(|\langle q_0|\psi\rangle|^2\) and \(|\langle q_0|\psi\rangle|^2\), \(\Delta q_0\) and \(\Delta p_0\), satisfy the uncertainty relation derived above. By definition,

\[
(\Delta q_0)^2 \equiv \int q_0^2 |\langle q_0|\psi\rangle|^2 dq_0 - \left( \int q_0 |\langle q_0|\psi\rangle|^2 dq_0 \right)^2,
\]

and

\[
(\Delta p_0)^2 \equiv \int p_0^2 |\langle p_0|\psi\rangle|^2 dp_0 - \left( \int p_0 |\langle p_0|\psi\rangle|^2 dp_0 \right)^2.
\]

By direct substitution, we can obtain

\[
\Delta q_0 \Delta p_0 \geq \frac{\hbar}{2}.
\]

In general, the eigenstates of \(p_0\) are not the eigenstates of the Hamiltonian, or the energies of the system; it is only in the special case where \(|\psi\rangle\) is a solution to the Schrödinger equation that \(\Delta p_0 = \Delta E\). Whereas the energy eigenstates for a bounded system are discrete, it is not possible to directly relate the constraints of the equation \(\Delta p_0 \Delta E > \frac{\hbar}{2}\) to Fourier transform pairs.
4 Time of Arrival in Quantum Mechanics

4.1 Introduction

The problem of arrival time in quantum mechanics has been intensively investigated [3] [10]. We believe that a quantum formalism where time and space are on the equal footing can lead to a simple solution to this problem. The problem to be solved can be succinctly stated as follows. If we have a source located at \( x = x_0 \) that produces a particle with a specific wave vector \( |\psi_0\rangle \) at time \( t = t_0 \), what is the probability that a detector at position \( X \) finds the particle at time \( t \), which we denote as the “time of arrival”, or “arrival time”?

Classically, the arrival time can be determined by inverting with respect to time the solution to the equations of motion corresponding to the initial position and momentum. For example, consider a non-relativistic free particle of which the dynamics is generated by the Hamiltonian \( H = \frac{p^2}{2m} \). Let \( x(t; x_0, p_0) = \frac{p_0}{m} t + x_0 \) be the general solution to equations of motion corresponding to the initial position \( x_0 \) and momentum \( p_0 \) at time \( t_0 \). Inverting this with respect to \( t \) yields the arrival time \( T \) at which the particle is detected at position \( X \),

\[
T(X) = t(X; x_0, p_0) = \frac{m(X - x_0)}{p_0}.
\]

In quantum mechanics, however, the problem is rather subtle, since any detecting or measuring will inevitably affect the wave function. Thus, we can only determine the time of arrival probabilistically.

Using the completeness of the states \( |t, x\rangle \) we can write the inner product of an arbitrary state \( |\psi\rangle \) as

\[
\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \psi | t, x \rangle \langle t, x | \psi \rangle dt dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x, t)|^2 dt dx . \tag{4.1}
\]
We interpret Eq. (4.1) in terms of a probability as $|\psi(x,t)|^2dt\,dx$ being the probability of measuring the particle in a small region of spacetime. More specifically, the probability that the particle is in the region of spacetime between $x$ and $x + dx$ and $t$ and $t + dt$ is

$$\rho(x,t)dt\,dx = \frac{|\psi(x,t)|^2dt\,dx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x,t)|^2dt\,dx}.$$  \hfill (4.2)

Where $\rho(x,t)$ is the probability density and the normalization is such that the probability over all of spacetime is unity:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x,t)dt\,dx = \lim_{T \to \infty} \int_{-\infty}^{\infty} \int_{-T}^{T} \int_{-\infty}^{\infty} |\psi(x,t)|^2dt\,dx = 1.$$ \hfill (4.3)

Consider now a region of spacetime $A$ where $x_a - \frac{\varepsilon}{2} \leq x \leq x_a + \frac{\varepsilon}{2}$ and $-\infty < t < \infty$. Since we are free to choose the normalization, we choose it such that over this restricted region

$$\int_{-\infty}^{\infty} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} \rho_A(x,t)dx\,dt = \lim_{T \to \infty} \frac{\int_{-T}^{T} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2dx\,dt}{\int_{-T}^{T} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2dx\,dt} = 1.$$ \hfill (4.4)

Where in the limit where $T$ goes to infinity, $\rho_A(x,t)$ is the probability density in this restricted region

$$\rho_A(x,t) = \frac{|\psi(x,t)|^2}{\int_{-\infty}^{\infty} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2dx\,dt},$$ \hfill (4.5)

and $\rho_A(x,t)dx\,dt$ is the probability that within the restricted spacetime region $A$ the particle is between $x$ and $x + dx$ and $t$ and $t + dt$. For this region of spacetime $A$, the average time is

$$\bar{t} = \int_{-\infty}^{\infty} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} t \rho_A(x,t)dx\,dt = \frac{\int_{-\infty}^{\infty} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} t |\psi(x,t)|^2dx\,dt}{\int_{-\infty}^{\infty} \int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} |\psi(x,t)|^2dx\,dt}.$$ \hfill (4.6)
For \( \varepsilon \ll 1 \) we can write for a smooth function \( f(x) \),

\[
\int_{x_a - \frac{\varepsilon}{2}}^{x_a + \frac{\varepsilon}{2}} f(x) \, dx = f(x_a) \varepsilon .
\]

(4.7)

So \( \bar{t} \) in Eq. (4.6) becomes

\[
\bar{t} = \frac{\int_{-\infty}^{\infty} t |\psi(x_a, t)|^2 \, dt}{\int_{-\infty}^{\infty} |\psi(x_a, t)|^2 \, dt} .
\]

(4.8)

In a similar way, the average position for a region of spacetime \( B \) where \( t_b - \frac{\varepsilon}{2} \leq t \leq t_b + \frac{\varepsilon}{2} \) and \(-\infty < x < \infty \) is

\[
\bar{x} = \int_{-\infty}^{\infty} \int_{t_b - \frac{\varepsilon}{2}}^{t_b + \frac{\varepsilon}{2}} x \rho_B(x, t) \, dx \, dt = \frac{\int_{-\infty}^{\infty} \int_{t_b - \frac{\varepsilon}{2}}^{t_b + \frac{\varepsilon}{2}} x |\psi(x, t)|^2 \, dx \, dt}{\int_{-\infty}^{\infty} \int_{t_b - \frac{\varepsilon}{2}}^{t_b + \frac{\varepsilon}{2}} |\psi(x, t)|^2 \, dx \, dt} ,
\]

(4.9)

where

\[
\rho_B(x, t) = \frac{|\psi(x, t)|^2}{\int_{-\infty}^{\infty} \int_{t_b - \frac{\varepsilon}{2}}^{t_b + \frac{\varepsilon}{2}} |\psi(x, t)|^2 \, dx \, dt} ,
\]

(4.10)

and \( \rho_B(x, t) \, dx \, dt \) is the probability that within the restricted spacetime region \( B \) the particle is between \( x \) and \( x + dx \) and \( t \) and \( t + dt \). Making use of Eq. (4.7) again, Eq. (4.9) becomes

\[
\bar{x} = \frac{\int_{-\infty}^{\infty} x \varepsilon |\psi(x, t_b)|^2 \, dx}{\int_{-\infty}^{\infty} \varepsilon |\psi(x, t_b)|^2 \, dx} = \frac{\int_{-\infty}^{\infty} x |\psi(x, t_b)|^2 \, dx}{\int_{-\infty}^{\infty} |\psi(x, t_b)|^2 \, dx} .
\]

(4.11)

Many attempts have been made to find a desirable solution to the arrival time problem. Among such efforts is the construction of a “time-of-arrival” operator. However, several concerns are raised with this approach [10]. As Allcock [11] has noted, the construction of these operators often does not take place in the Hilbert Space, which makes many conventional techniques in quantum mechanics inapplicable. Even in the case where one is able to construct the operator within Hilbert Space, it usually involves complex procedures such as decomposing the particle’s Heisenberg states into the eigenstates of this operator [10].

We believe that our formalism that treats time and space equally can provide a straight-
forward answer to the problem. We will start with the one-dimensional free particle case used in Grot’s paper, and demonstrate that directly applying Eq. (4.8) yields what we call the “presence time”; the results of which, as we will show, agree with what Grot’s. Nevertheless, the notion of “presence time” is not equivalent to arrival time. In the second subsection, we distinguish between the two notions, and show we approach to the arrival time. Specifically, we present a method to calculate the affected wave function after each time the detector makes a measurement. The third subsection will be a discussion, in which we explore features of this method and compare the results with the classical predictions.

4.2 Presence Time of a Free Particle

We consider a nonrelativistic free particle with a Gaussian wave packet localized at a point \( x_0 \) when \( t = 0 \) and moving to the right. Let this wave packet be given by the normalized solution to the Schrödinger equation \[10\],

\[
\psi(x,t) = \left( \frac{\delta^2}{2\pi} \right)^{1/4} \frac{\exp(-k_0^2\delta^2)}{\sqrt{\delta^2 + \frac{it\hbar}{2m}}} \exp \left( \frac{[2\delta^2k_0 + i(x - x_0)]^2}{4\delta^2 + \frac{2it\hbar}{m}} \right),
\] (4.12)

where \( k_0 \) is the wave number, \( \delta \) is the width of the wave packet at time \( t = 0 \), \( m \) is the mass of the particle, and \( x \) is the position where the particle is detected at \( t \). Expectation values are calculated \[10\] to be

\[
\langle p(t) \rangle = \hbar k_0, \quad \langle \Delta p(t) \rangle = \frac{\hbar}{2\delta},
\] (4.13)

\[
\langle x(t) \rangle = x_0 + \hbar k_0 t/m, \quad \langle \Delta x(t) \rangle = \delta \sqrt{1 + \frac{t^2\hbar^2}{4\delta^4m^2}}.
\] (4.14)

We now apply Eq. (4.8) to calculate the arrival time. Before we start, however, we need to consider the experimental setting. Since Eq. (4.8) does not take into account perturbations of the wavefunction due to the measuring device when the particle is not detected, it requires us
to make detections without disturbing the wavefunction. The only way that would guarantee
an undisturbed wave function is to insert our detector at time $t_1$ and measure only once. If
we do not detect the particle, we take the detector out, redo the experiment and insert insert
the detector at $t_2$. We repeat this until the particle is detected, and the average time at which
the particle is detected is the presence time. Theoretically we would repeat this procedure an
infinite number of times, before we can assume continuity. To be consistent with Grot, we
take the following values for constants $x_0, k_0, \delta$ and $h/m$:

$$x_0 = -5, \quad k_0 = 20, \quad \delta = 0.5, \quad h/m = 1,$$

and we consider the probability densities for six cases: $X = -5, -3, -1, 1, 3, 5$. When $x$ is
fixed, corresponding to a spacetime region $\mathcal{A}$, we will use Eq. (4.8) to write a probability
density as

$$\pi_t(X, t) = \frac{|\psi(X, t)|^2}{\int_{-\infty}^{\infty} |\psi(X, t)|^2 dt}.$$  \hspace{1cm} (4.15)

For fixed $x$, the product

$$P(x, t) = \pi_t(x, t)dt$$  \hspace{1cm} (4.16)

is the probability that the particle has a time coordinate between $t$ and $t + dt$ and position
in the infinitesimal neighborhood between $x - \frac{\varepsilon}{2}$ and $x + \frac{\varepsilon}{2}$ for small $\varepsilon$. Due to the complex
nature of $\psi$ and the integration, we use Mathematica for numerical evaluations. The results
are shown in Fig. (1), plotted against time for each of the six $X$ values. We find that this
plot is in agreement with what Grot obtains [10]. The probability density function $\pi_t$ is
concentrated around the classical trajectory of the particle $x = x_0 + p_0 \frac{T}{m}$, but it also has a
“spread” in time. The average of presence times at $x = X$ is just the expectation value of $t$,

$$\bar{t} = \langle t \rangle = \int_{-\infty}^{\infty} t \pi_t(X, t) \, dt,$$  \hspace{1cm} (4.17)
Figure 1: Probability densities $\Pi_i(x, t)$ plotted against $t$ for the $i$-th value taken from $X = -5, -3, -1, 1, 3, 5$.

and we find the average values are

$$\bar{\tau}_1 \to 0, \quad \bar{\tau}_2 \cong 0.101, \quad \bar{\tau}_3 \cong 0.201, \quad \bar{\tau}_4 \cong 0.302, \quad \bar{\tau}_5 \cong 0.402, \quad \bar{\tau}_6 \cong 0.503$$

corresponding to the six $X$ values.

For the purpose of comparison, we take a look at the probability density at fixed $t$, corresponding to a spacetime region $B$, we will use Eq. (4.10) to write a probability density as

$$\pi_x(x, \tau) = \frac{|\psi(x, \tau)|^2}{\int_{-\infty}^{\infty} |\psi(x, \tau)|^2 dx}, \quad (4.18)$$

which is the usual Schrödinger probability density for position at time $\tau$. In Fig. (2) the Schrödinger probability distributions are plotted against position $x$ at each of the six $t_i$ values taken. The positions for the peaks of each Schrödinger probability density are indeed the same as the six $X$ values.
Figure 2: Schrödinger probability densities $\Pi_i(x, t_i)$ plotted against $x$ at each of the times taken from $t_i = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.

4.3 The Arrival Time of a Free Particle

Nevertheless, the average of presence time that we found in the last subsection is not arrival time, which requires a different experimental setting. Again, we use Eq. (4.12) as the initial wave function at $t = 0$. Suppose we make all detections at $x_c$. At $t_1$ we insert our detector to make the first measurement. The probability that the particle is in the spacetime region between $x_c, x_c + dx$ and $t_1, t_1 + dt$ is from Eq. (4.16)

$$P_1(x_c, t_1) = \pi_t(x_c, t_1)dt = \frac{\int_{-\infty}^{\infty} |\psi_0(x_c, t_1)|^2 dt}{\int_{-\infty}^{\infty} |\psi_0(x_c, t)|^2 dt}. \quad (4.19)$$

On the other hand, if we did not detect the particle at $x_c, t_1$, we leave the detector there to continue detection at $t_2, t_3, t_4...$, each of the $t$’s is of a small increase to the previous time,
until detection is made. However, the previous measurement at \((x_c, t_1)\) has affected the wave function. We can obtain the affected wave function as follows. Since the wave function \(\psi_0(x, t)\) can be viewed as a distribution along the position axis at time \(t_0\), we can propagate from \(t_0\) to \(t_1\) to get \(\psi_1(x, t)\). Of course, if there were no disturbance, the resulted wave function is just the original function \(\psi_0(x, t)\). But since the particle is absent at \((x_c, t_1)\), the distribution at this point must be omitted. The omitted distribution can be obtained by propagating \(\psi_0\) at \((x_c, t_1)\) to \((x_c, t_2)\) using a propagator \(K(x, x', t, t_1)\), and then integrated over the width of the detector. Subtract this portion from \(\psi_0\), we obtain the affected wave function. Suppose the detector spans from \(-\frac{\epsilon}{2}\) to \(\frac{\epsilon}{2}\) where \(\epsilon\) is a small number, and the center of detector is placed at \(x_c\). We write the original wave function \(\psi_0(x, t)\) as the following,

\[
\psi_0(x, t) = \int_{-\infty}^{\infty} K(x, x_c, t, t_1)\psi_0(x_c, t_1)dx_c
\]

\[
= \int_{\infty}^{x_c - \frac{\epsilon}{2}} K(x, x', t, t_1)\psi_0(x', t_1)dx' + \int_{x_c + \frac{\epsilon}{2}}^{\infty} K(x_c, x', t, t_1)\psi_0(x', t_1)dx'
\]

\[
+ \int_{x_c - \frac{\epsilon}{2}}^{x_c + \frac{\epsilon}{2}} K(x, x', t, t_1)\psi_0(x', t_1)dx'.
\]

To get the affected wave function at \(t_1\), we simply move the last term on the right hand side to the left,

\[
\int_{\infty}^{x_c - \frac{\epsilon}{2}} K(x, x', t, t_1)\psi_0(x', t_1)dx' + \int_{x_c + \frac{\epsilon}{2}}^{\infty} K(x_c, x', t, t_1)\psi_0(x', t_1)dx'
\]

\[
= \psi_0(x, t) - \int_{x_c - \frac{\epsilon}{2}}^{x_c + \frac{\epsilon}{2}} K(x, x', t, t_1)\psi_0(x', t_1)dx'
\]

\[
\equiv \psi_1(x, t).
\]

In general, the affected wave function can be calculated the previous wave function at an earlier time, as

\[
\psi_i(x, t) = \psi_{i-1} - \int_{x_c - \frac{\epsilon}{2}}^{x_c + \frac{\epsilon}{2}} K(x, x', t, t_i)\psi_{i-1}(x', t_i)dx'. \quad (4.20)
\]
Analogous to Eq. (4.19) we write

\[ P_i(x, t_i) = \pi_i(x, t_i) dt = \int_{-\infty}^{\infty} |\psi_{i-1}(x, t)|^2 dt \]  

for the \( i \)th measurement.

So far, we have taken care of the affected wavefunction; now we need to reconsider the probabilistic approach that we have used previously in the presence time case. We use a simple analogy of throwing dice here to illustrate the difference between the presence time case and the arrival time case. If we throw a dice once, the probability of getting a one is 1/6. This is analogous to the presence time, in the sense that if we measure once at \( t_1 \), the probability of getting the particle is just \( P_1 \). If we throw the dice twice, the probability that we get a one in the second turn but not in the first is (5/6 * 1/6). If I throw it \( n \) times and do not get a one until the \( n \)-th time, the probability becomes \((5/6)^{n-1} \ast (1/6) \) This is analogous to the arrival time probability, which we denote as \( \Pi_i(x, t_i) \), in the sense that the probability that we do not detect the particle at \( x \) until the \( i \)th time, or \( t_i \) is \( \Pi_i(x, t_i) \), and should be the product of every probability that we do not detect the particle in each one of the \((i - 1)\) trials, multiplied by the probability of detecting the particle in the \( i \)th trial:

\[ \Pi_i(x, t_i) = \prod_{j=1}^{i-1} [1 - P_j(x, t_j)] P_i(x, t_i). \]  

Finally, we consider the time of arrival. We repeat the experiment until we detect the particle at \( t_i \). The time of arrival \( \bar{t} \) is the average time that we would expect to detect the particle if we perform the experiment by making up to \( n \) detections is

\[ \bar{t} = \frac{\sum_{i=1}^{n} t_i \ast \Pi_i}{\sum_{i=1}^{n} \Pi_i}. \]  

(4.23)


4.4 Discussion

We now discuss some details of the above method. First, we calculate the arrival time for the free particle example that we have been considering throughout the preceding subsections. We suppose the detector has a width of 0.02, which is much smaller than the width of the wave packet, and set the center of it to be at \( x_c = -3 \). The propagator for the free particle case is

\[
K(x, x_c, t, t_1) = \frac{1}{\sqrt{2\pi i(t - t_1)}} e^{\frac{i(x - x_c)^2}{2(t - t_1)}}
\]  

(4.24)

where \( t_1 \) is the time we insert the detector to observe. We take a time interval from \(-0.02\) to \(0.22\) and evaluate ten points evenly spread out within this interval. See Appendix A for the details of the Mathematica program. Fig. (3) shows the probability density \( \Pi \) calculated using Eq. (4.22) at each time taken within the interval. We then calculate the arrival time using Eq. (4.23). The result is \( \bar{t} \approx 0.083 \).

![Figure 3: Probability densities \( \Pi(x,t) \) plotted against \( t \) for a detector of small width placed at \( x = -3 \), which measures every 0.024 unit of time starting from -0.02 to 0.022. This interval ensures that the probability density distribution can be properly centered, and the starting point is chosen to be a little variation off zero to avoid singularity at zero.](image-url)

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Fig.(3) has demonstrated a peak located around 0.083 and a skewed pattern toward the left side. On right side of the peak the probability density reduces very rapidly. This indicates that the longer the detector has waited, the more unlikely it will detect the particle. Moreover,

![Figure 4](image_url)

**Figure 4**: Comparison between the probability density \( \Pi_t(x = 3, t) \) by an unchanging wave function \( \psi_0(x, t) \) (squares) verses that by wave functions that are continuously affected (dots). Both of the probability densities are calculated using Eq. (4.22).

the peak is located at an earlier time than the presence time that we obtained from Eq. (4.17), which is roughly 0.1. Both the recurring reduction in the affected wave function and the way we calculate the probability density contribute to this phenomenon. Fig.(4) compares the probability density plot with both contributions to the one with only the probabilistic factor. Clearly, changing in wave function does not shift the peak or the distribution very much. Thus the shifting of arrival time must be attributed to the way we calculate the probability density.

On the other hand, changing the wave function is responsible for the drastic oscillation of the distribution. Due to the limited computing power, we have to calculate the distribution discretely, and the time division could only run up to 10 points. Both make it rather hard to identify the effects of the changing wave functions from the plot. Moreover, using a detector
with a finite width has canceled out a great portion of the oscillations introduced by the propagator. Albeit these difficulties, in Fig.(4) we can still see that the dots (with changing wave functions) are varying more radically than the squares (with unchanged wave function).

To further demonstrate how the probabilistic factor shifts the arrival time to the left, we now compare the ten-point case under consistent wave function at an extreme case of 5000 points, which is shown in Fig.(5). The probability density distribution is shifted to 0.019, while the shape of the distribution remains a smooth Gaussian. Thus we have confirmed that the probabilistic factor shifts the arrival time, while the affected wave functions attributes to the oscillation of the probability density distribution.

![Figure 5: Probability density Π_t(x = 3, t) plots for n = 5000 and n = 10 under an unaffected wave function.](image)

We now vary the width of the detector and study its effect. As seen in Fig.(2), the width of the wave packet is between three to four units. We have set ϵ = 0.01; the total width of the detector, 0.02, is much smaller than that of the wave packet. We now consider the case where ϵ = 1.0 and evaluate the arrival time for the same particle, and we obtain the result to be 0.12, which is much later than the values obtained classically.

Fig.(6) shows a comparison between the probability densities of the small-width detector
case and the large-width detector case. Compared to the small-width case, the probability
densities for the large-width detector have much less intensity at the peak area, but toward the
tails they die out more slowly. This is as expected, because the larger the detector is relative
to the wave packet, the lesser oscillation the propagator will be able to generate. Another
interesting observation is that technically speaking, the entire wave packet is “inside” the
detector, which could be responsible for the longer arrival time and a flatter distribution.
However, the causes for a longer arrival time is still unclear here. Tests on an extreme case
with a greater amount of points are needed, if the facilities allow.

Figure 6: Comparison between the probability density $\Pi_t(x = 3, t)$ for the small-width detector
case (dots) and the large-width detector (squares).
5 Summary and Conclusion

In this project, we outlined a new quantum formalism that treats time and space on an equal footing. The key to this formalism, the commutation relations, emerged naturally from the extended theories of Hamiltonian Mechanics. Canonical quantization leads to for space and time translations, and a corresponding spacetime representation was developed. We then derived the Schrödinger equation and the propagator as natural products of the representation, and recovered the energy-time uncertainty relation.

In the second part of the project, we focused on an application of the arrival time problem in quantum mechanics. Although various solutions have been given on this subject, most of them involve greater complexities than our treatment. Under the idea to treat time as a dynamical variable, we were able to calculate the probability of arrival time directly by integrating the probability density over time. Specifically, we distinguished arrival time from the presence time by introducing a statistical concept into the calculation of probability density, and have obtained many interesting results.
References


Appendix A. Calculations of the Arrival Time and Probability Densities for the Free Particle Case

1. The Arrival Time Calculation

- The Basic Setup

We use the same wave function as we used in Appendix A. with the same constants.

\[
\begin{align*}
k &= 20; \\
x_0 &= -5; \\
del &= .5; \\
\psi_0(x_0, t) &= \frac{(\text{del}^2)}{2\pi} \times \frac{\text{Exp}[ - k^2 \times \text{del}^2 ]}{\sqrt{\text{del}^2 + \frac{it}{2}}} \times \frac{\text{Exp}\left[\frac{2 \times \text{del}^2 \times k + i \times (x - x_0)}{4 \times \text{del}^2 + 2 \times i \times t}\right]}{A[0]}
\end{align*}
\]

We set width of the detector to be from -ep to ep, and the center of it to be at x_p.

\[
\begin{align*}
x_p &= -3; \\
ep &= .01;
\end{align*}
\]

We consider the time interval from t_i to t_f, and take n points evenly spread out within in the interval. The starting point was purposely chosen to be a little variation off zero, in order to avoid the singular point at zero. Another advantage of this choice is that the graph could be properly centered. The notion tm[i] represents the i-th point within the interval. Due to the complex programming, Mathematical is only capable to carry out calculations for about 10 divisions of the entire time interval. A more powerful computing device is needed to carry our further divisions.

\[
\begin{align*}
t_i &= -0.02; \\
t_f &= 0.22; \\
n &= 11; \\
dt &= (t_f - t_i) / (n - 1); \\
tm[1] &= t_i; \\
Do[tm[i] = tm[i-1] + dt, \{i, 2, n\}]
\end{align*}
\]

- Calculating Affected Wave Function

If we detect the particle the first time we measure it (at tm[1]) at position ep, then we can calculate the probability that the particle is at (tm[1], x_p) to be p[1] below, where A[0] is the normalization constant for the zero-th wave function. The probability density prob[1] is the same as p[1].
A[0] = NIntegrate[Abs[psi[0, x, tm[1]]]^2, {x, -9, 9}];
p[1] = (1/A[0]) * Abs[psi[0, xp, tm[1]]]^2;
prob[1] = p[1];

Suppose, however, we detect that the particle is absent at (ep, t1). If this point were not omitted from the entire wave distribution, we could have propagate its wave function distribution by using the propagator K below.

\[
K[x, xc, t, to] := (2 \pi i t)^{-1/2} \exp\left[\frac{i (x - xc)^2}{2 (t - to)}\right]
\]

The affected wave function is then calculated to be

\[
\text{Do}[\psi[i, x, t] := \psi[i-1, x, t] - \text{NIntegrate}[K[x, xc, t, tm[i]] \cdot \psi[i-1, xc, tm[i]], \{xc, xp-ep, xp+ep\}, \text{AccuracyGoal} \to 20], \{i, 1, n-1\}]
\]

We interpolate the function and normalize the function as follows.

\[
\text{Do}[f1 = \text{Table}[\{x, \text{Abs}[\psi[i, x, tm[i+1]]]^2\}, \{x, -9, 9, 1\}]; 
f2 = \text{Interpolation}[f1]; 
A[i] = \text{NIntegrate}[f2[x], \{x, -9, 9\}, \text{AccuracyGoal} \to 20], \{i, 1, n-1\}]
\]

- **Probability and Probability Density**

As before, the probability p[i] represent that the particle is detected at the i-th trial, or at time tm[i]. The probability density, prob[i], refers to the event that the particle has not been detected through all the previous trials and is detected at this one.

\[
\text{Do}[p[i] = \frac{1}{A[i-1]} \cdot \text{Abs}[\psi[i-1, xp, tm[i]]]^2, \{i, 2, n\}]
\]

\[
\text{Do}[\text{prob}[i_] := (\prod_{k=1}^{i-1} (1-p[k])) \cdot p[i], \{i, 2, n, 1\}]
\]

Plotting the probability densities against time, we obtain:
A table for each of the points on the above graph is listed as follows

\[
\text{Table}\left[\{\text{tm}[i], \text{prob}[i]\}, \{i, n\}\right]
\]

\[
\{\{-0.02, 8.0633 \times 10^{-6}\}, \{0.004, 0.000506616\}, \\
\{0.028, 0.0129422\}, \{0.052, 0.129243\}, \{0.076, 0.448787\}, \\
\{0.1, 0.348161\}, \{0.124, 0.0354916\}, \{0.148, 0.00432712\}, \\
\{0.172, 0.00420129\}, \{0.196, 0.00029259\}, \{0.22, 3.9076 \times 10^{-6}\}\}
\]

### The Arrival Time

The arrival time, \(t_{\text{bar}}\), is calculated to be

\[
t_{\text{bar}} = \frac{\sum_{j=1}^{n} (\text{tm}[j] \times \text{prob}[j])}{\sum_{j=1}^{n} \text{prob}[j]}
\]

0.0827915

### 2. Variation with the Width of Detector

In the previous case we have a detector of small width compare to the width of the wave packet. We now study the opposite case where the detector's width is larger than the wave packet.

\[
k = 20; \\
x_0 = -5; \\
del = .5;
\]

\[
\psi_0[x, x_0, t_] := \left(\frac{\text{del}^2}{2\pi}\right)^{1/4} \cdot \frac{\exp[-k^2 \cdot \text{del}^2]}{\sqrt{\text{del}^2 + \frac{\text{i}^2 t}{2}}} \cdot \frac{\exp\left[\frac{2 \cdot \text{del}^2 \cdot k + \text{i} \cdot (x - x_0)^2}{4 \cdot \text{del}^2 + 2 \cdot \text{i} \cdot t}\right]}{4 \cdot \text{del}^2 + 2 \cdot \text{i} \cdot t}
\]
We set width of the detector to be from -2.5 to 2.5, so that the total width is greater than the width of the wave packet.

\[
\begin{align*}
xp &= -3;  \\
ep &= 2.5;  \\
ti &= -0.02;  \\
tf &= 0.22;  \\
n &= 11;  \\
dt &= (tf - ti) / (n - 1);  \\
tm[1] &= ti; \\
\text{Do}[tm[i] = tm[i - 1] + dt, \{i, 2, n\}] \\
\end{align*}
\]

- **Calculating Affected Wave Function**

\[
\begin{align*}
A[0] &= \text{NIntegrate}\left[\text{Abs}[\psi[0, x, tm[1]]]^2, \{x, -9, 9\}\right];  \\
p[1] &= (1 / A[0]) * \text{Abs}[\psi[0, xp, tm[1]]]^2;  \\
\text{prob}[1] &= p[1];  \\
K[x_-, xc\_\:, t\_\:, to\_\:] &= (2 \pi \pi * i * (t - to))^{(1/2)} * \text{Exp}\left[\frac{i * (x - xc)^2}{2 * (t - to)}\right]
\end{align*}
\]

The affected wave function is then calculated to be

\[
\begin{align*}
\text{Do}[\psi[i\_, x\_, t\_] &= \psi[i - 1, x, t] - \\
\text{NIntegrate}[K[x, xc, t, tm[i]] * \psi[i - 1, xc, tm[i]],  \\
(xc, xp - ep, xp + ep), \text{AccuracyGoal} \rightarrow 20 ], \{i, 1, n - 1\}]
\end{align*}
\]

\[
\begin{align*}
\text{Do}[f1 &= \text{Table}[\{x, \text{Abs}[\psi[i, x, tm[i + 1]]]^2\}, \{x, -9, 9, 0.1\}];  \\
f2 &= \text{Interpolation}[f1];  \\
A[i] &= \text{NIntegrate}[f2[x], \{x, -9, 9\}, \text{AccuracyGoal} \rightarrow 20 ], \{i, 1, n - 1\}]
\end{align*}
\]

- **Probability and Probability Density**

\[
\begin{align*}
\text{Do}[p[i] &= \frac{1}{A[i - 1]} * \text{Abs}[\psi[i - 1, xp, tm[i]]]^2, \{i, 2, n\}]
\end{align*}
\]

\[
\begin{align*}
\text{Do}[\text{prob}[i\_] &= \left(\prod_{k=1}^{i-1} (1 - p[k])\right) * p[i], \{i, 2, n, 1\}]
\end{align*}
\]

Plotting the probability densities against time, we obtain:
The Arrival Time

The arrival time, \( t_{\text{bar}} \), is calculated to be

\[
\bar{t} = \frac{\sum_{j=1}^{n} (tm[j] \cdot \text{prob}[j])}{\sum_{j=1}^{n} \text{prob}[j]}
\]

\( 0.117401 \)
Comparison

Tablesmall = {{-0.02`, 8.063295956602108`*^-6},
              {0.004`, 0.0005066156989562808`}, {0.028`, 0.012942200647418736`},
              {0.052000000000000001`, 0.4487866405175539`},
              {0.1`, 0.348160745411065`}, {0.124`, 0.03549159360983835`},
              {0.148`, 0.004327115917659222`}, {0.172`, 0.0004201294381782778`},
              {0.19599999999999998`, 0.00029259035870985035`},
              {0.21999999999999997`, 3.9075997140257524`*^-6}};
Tablebig = {{-0.02`, 8.063295956602108`*^-6}, {0.0066666666666666645`,
            0.0004034170820417378`}, {0.0333333333333333326`, 0.011788841842369781`},
            {0.05999999999999999`, 0.1071566299259171`}, {0.08666666666666666`,
            0.32568014409730506`}, {0.11333333333333333`, 0.2434245988464698`},
            {0.13999999999999999`, 0.03459661797326328`},
            {0.16666666666666666`, 0.018744539634118772`},
            {0.19333333333333333`, 0.04138659859857544`},
            {0.22`, 0.051719017771575004`}};
ListPlot[{Tablesmall, Tablebig},
          PlotStyle -> {Orange, Gray}, PlotMarkers -> {Automatic, 16},
          AxesLabel -> {Style["t"], Italic, 20}, Style["\(\pi(xp,t)\)", Italic, 20]},
          Filling -> Axis, LabelStyle -> Directive[Bold],
          TickStyle -> Directive[Black, Thick, 16]}
Appendix B. Comparison: Case of Consistent Wave Function

1. The Case of Consistent Wave Function

■ The Basic Setup

We use the same function and setup as in the case where the wave function will be affected.

\[
\begin{align*}
k &= 20; \\
x_0 &= -5; \\
\delta &= 0.5; \\
\psi[i_-, x_-, t_] &= \left(\frac{\delta^2}{2 \pi}\right)^{1/4} \cdot \frac{\exp[-k^2 \cdot \delta^2]}{\sqrt{\delta^2 + \frac{i \cdot t}{2}}} \\
&\quad \cdot \exp\left[\frac{2 \cdot \delta^2 \cdot k + i \cdot (x - x_0)^2}{4 \cdot \delta^2 + 2 \cdot i \cdot t}\right]
\end{align*}
\]

Detector has width from \(-\epsilon_p\) to \(\epsilon_p\), and is centered at \(x_p\).

\[
\begin{align*}
x_p &= -3; \\
\epsilon_p &= 0.01;
\end{align*}
\]

We consider the time interval from \(t_i\) to \(t_f\), and take \(n\) points evenly spread out within the interval. The starting point was purposely chosen to be a little variation off zero, in order to avoid the singular point at zero. Another advantage of this choice is that the graph could be properly centered. The notion \(tm[i]\) represents the \(i\)-th point within the interval. Due to the complex programming, Mathematical is only capable to carry out calculations for about 10 divisions of the entire time interval. A more powerful computing device is needed to carry our further divisions.

\[
\begin{align*}
t_i &= -0.02; \\
t_f &= 0.22; \\
n &= 11; \\
dt &= (t_f - t_i) / (n - 1); \\
tm[1] &= t_i; \\
Do[tm[i] = tm[i - 1] + dt, \{i, 2, n\}]
\end{align*}
\]

■ Probability and Probability Densities

If we detect the particle the first time we measure it at \(tm[1]\) and at position \(\epsilon_p\), the probability \(p[i]\) and the probability density \(prob[i]\) are

\[
\begin{align*}
A[0] &= \text{NIntegrate}[Abs[\psi[0, x, tm[1]]]^2, \{x, -9, 9\}]; \\
p[1] &= \left(1 / A[0]\right) \cdot Abs[\psi[0, x_p, tm[1]]]^2; \\
prob[1] &= p[1];
\end{align*}
\]

If we did not detect the particle at \((\epsilon_p, t_1)\), and assume the wave function is not affected by our measurement, we have
Do[A[i] = NIntegrate[Abs[psi[i, x, tm[i + 1]]]^2, {x, -9, 9}, AccuracyGoal -> 20], {i, 1, n - 1}]
Do[p[i] = \[frac{1}{A[i - 1]}\] * Abs[psi[i - 1, xp, tm[i]]]^2, {i, 2, n}]
Do[prob[i_] := (\[prod\]_{k=1}^{i-1} (1 - p[k])) \[times\] p[i], {i, 2, n, 1}]

Plotting the probability densities against time:

\[
\text{Plot2} = \text{ListPlot}[[\text{Table}[\{tm[i], prob[i]\}, \{i, n\}]], \\
\quad \text{Filling} \rightarrow \text{Axis}, \text{PlotStyle} \rightarrow \{\text{Red}\}, \text{PlotMarkers} \rightarrow \{\text{Automatic}, 16\}, \\
\quad \text{AxesLabel} \rightarrow \{\text{Style["t"}, \text{Italic, 20}], \text{Style["\pi(xp,t)"}, \text{Italic, 20}], \\
\quad \text{LabelStyle} \rightarrow \text{Directive[Bold], TicksStyle} \rightarrow \text{Directive[Black, Thick, 16]}\]

A table for each of the points on the above graph is listed as follows

\[
\text{Table2} = \text{Table}[\{tm[i], prob[i]\}, \{i, n\}] \\
\quad \{\{-0.02, 8.0633 \times 10^{-6}\}, \{0.004, 0.000501373\}, \\
\quad \{0.028, 0.0127516\}, \{0.052, 0.126439\}, \{0.076, 0.43254\}, \\
\quad \{0.1, 0.334675\}, \{0.124, 0.0467014\}, \{0.148, 0.00651716\}, \\
\quad \{0.172, 0.000737455\}, \{0.196, 0.0000487439\}, \{0.22, 1.83604 \times 10^{-6}\}\}
\]

■ The Arrival Time

The arrival time, tbar, is calculated to be

\[
tbar = \frac{\sum_{j=1}^{n} (tm[j] \times prob[j])}{\sum_{j=1}^{n} prob[j]}
\]

0.0834268
2. Comparison

We now compare this case with the previous case where the wave function continues to be affected by the measurement. We plot the probability density distributions for both in the same graph.

\[ \text{Probability density distribution for both in the same graph.} \]

\[ \text{ListPlot[\{Table1, Table2\], Filling\rightarrow\text{Axis},} \]
\[ \text{PlotStyle\rightarrow\{Blue, Red\], PlotMarkers\rightarrow\{Automatic, 16\],} \]
\[ \text{AxesLabel\rightarrow\{Style\["t\], Italic, 20\], Style\["\pi(x_p, t)\], Italic, 20\]}\]
\[ \text{LabelStyle\rightarrow\{Directive\[Bold\], TicksStyle\rightarrow\{Directive\[Black, Thick, 16\}\]]} \]

![Graph showing probability density distributions](comparison2.nb)

The figure shows that the affected wave function is mainly responsible for the shape of the distribution, but does not shift the peak very much. The probabilistic factor, on the other hand, is mostly responsible for the inward shift of the arrival time.

2. The Extreme Case

We now consider the extreme case where n=500, and see how far it will shift the arrival time leftward.

\[ n = 5000; \]
\[ dt = (tf - ti) / (n - 1); \]
\[ tm[1] = ti; \]
\[ Do[tm[i] = tm[i - 1] + dt, \{i, 2, n\}] \]
\[ A[0] = NIntegrate[Abs[psi[0, x, tm[1]]]^2, \{x, -9, 9\}]; \]
\[ p[1] = (1/A[0]) * Abs[psi[0, xp, tm[1]]]^2; \]
\[ prob[1] = p[1]; \]
Do[A[i] = NIntegrate[Abs[psi[i, x, tm[i + 1]]]^2, {x, -9, 9}, AccuracyGoal -> 20], {i, 1, n - 1}]

Do[p[i] = 1/A[i - 1] * Abs[psi[i - 1, xp, tm[i]]]^2, {i, 2, n}]

Do[prob[i_] := (Product[1 - p[k], {k, 1, i - 1}]) * p[i], {i, 2, n, 1}]

Plot3 = ListPlot[Table[{tm[i], 277 * prob[i]}, {i, n}],
PlotRange -> All, PlotStyle -> {Pink, PointSize[Small]},
AxesLabel -> {Style"t", Italic, 20}, Style"\pi(xp,t)"", Italic, 20}],
LabelStyle -> Directive[Bold], TicksStyle -> Directive[Black, Thick, 16]]

\[ \pi(xp,t) \]

Table4 = {{-0.02`, 8.063295956602108`*^-6},
{0.004`, 0.000501373193576023`}, {0.028`, 0.012751562007506491`},
{0.052000000000000005`, 0.12643874769404856`},
{0.076000000000000001`, 0.43254027405389306`},
{0.1`, 0.33467520819781343`}, {0.124`, 0.04670143123958462`},
{0.148`, 0.0065171628791615924`}, {0.172`, 0.0007374546206114915`},
{0.19599999999999998`, 0.000048743913955518266`},
{0.21999999999999997`, 1.8360429853239033`*^-6}};
Show[Plot3, Plot4, PlotRange -> All]

\[ \pi(xp,t) \]

\[ tbar = \frac{\sum_{j=1}^{n} (tm[j] \cdot prob[j])}{\sum_{j=1}^{n} prob[j]} \]

0.0186655