The Hyperbolic Structure of the Complements of Rational Links with Conway Notation $mn$ for $m, n \geq 2$.

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Math 491: Independent Study in Hyperbolic 3-Manifolds
May 5, 2011

Abstract

We use a specific application of Adams algorithm to investigate the hyperbolic structure of rational links composed of two tangles. The investigation begins with the figure-eight knot (the simplest hyperbolic knot), and we note changes in the hyperbolic structure as half-twists are added to each of its component tangles. We model the complements of the links under investigation in a specific way such that the following patterns are observed. Our model for the complement of a link with Conway notation $mn$, $m, n \geq 2$, will contain $(m + n) - 3$ pairs of tetrahedra and $2((m + n) - 3)$ edge types. Furthermore, the tetrahedral configuration and edge equations corresponding to these models change in a predictable way as we each additional half-twist. The $2((m + n) - 3)$ edge equations are then simplified down to $(m + n) - 3$ equations and solved. Their solutions determine the hyperbolic structure and are used to determine the hyperbolic volume of each knot under investigation. As $m$ and $n$ increase, the growth rate of the hyperbolic volume decreases.
1 Constructing the Link

1.1 Knots and Tangles

In providing an informal definition of a knot, Adams advises us to “Take a string. Tie a knot in it. Now glue the two ends of the string together to form a knotted loop” (Figure 1). Once its ends are glued, the knotted string then cannot be unknotted without cutting it. Also, the string could not possibly pass through itself. Finally, if the string were made of rubber it could certainly be deformed (that is, stretched or contracted) without changing how the string is knotted. These properties are all satisfied under the formal definition of a knot. Typically, however, we think of a knot as having no thickness. Thus, Adams formally states that a knot is “a closed curve in space that does not intersect itself everywhere”.

![Figure 1: Tie the ends of a string together to obtain a knot.](image)

Although we can certainly imagine curves in space with no thickness, it is often more convenient to study knots using pictures of the knots called projections. Essentially, we can think of projections as snapshots of the knot from a particular viewpoint. Since there are infinitely many perspectives one may take and since knots are deformable, there are an infinite amount of acceptable projections that correspond to each knot. Also, while the knots themselves are closed curves nearly all projections appear as a set of disjoint curves with discontinuities where the knot would appear to cross itself. The areas of discontinuity are called crossings and represent points at which the knot crosses under
itself. Near each crossing we call the seemingly discontinuous arcs *understrands* while the connected arcs passing over them are called *overstrands*. Adams provides that projections with two or fewer crossings correspond to the trivial knot (or “unknot”), an unknotted, nonintersecting closed curve. Figure 2 contains a few commonly used projections of the figure-eight knot, an important example whose significance we will explain later.

![Figure 2: Three projections of the figure-eight knot.](image)

### 1.2 Rational Links

Next, Adams defines a *tangle* in a knot projection as “a region in the projection plane surrounded by a circle such that the knot...crosses the circle exactly four times”. Also, Adams dictates that we consider these four points of intersection as occurring in the compass directions NE, NW, SE and SW. In Figure 3 we isolate a tangle of the figure-eight knot.

![Figure 3: A tangle in the figure-eight knot.](image)
In our investigation, tangles are useful in that they can be independently composed and used to construct large families of higher-order structures called links. Essentially, a link is a collection of knots that may or may not be knotted with each other. In particular, a link is deemed to be nonsplittable if we cannot separate any of its component knots such that they lie on either side of a plane in 3-dimensional space. Also, note that knots are simply links with one component, and thus make up a subset of all links. The following process (as presented by Adams) demonstrates how we can use tangles to compose links.

We begin with the 0 tangle, defined as the tangle consisting of two unknotted, horizontal curves (Figure 4a). Imagine now that these curves are strings and that you are holding them in your hands such that both strings are pulled taught. Now, by twisting either hand toward your body you twist the strings such that they cross each other once and one lies over the other. Depending on which hand you use, we refer to these actions as putting a left- or right-handed twist into the tangle. Next, imagine that after putting a half-twist on the 0 tangle we think of the overstrand as a line segment. If we performed a left-handed half twist, the overstrand will have positive slope and we label the resulting tangle 1 (Figure 4b). Conversely, putting a right-handed twist on the 0 tangle produces the tangle \(-1\) (Figure 4c). Consequently, putting three left handed half-twists on the 0 tangle produces the 3 tangle (Figure 4d) and so on.

![Four tangles and their notations.](image)

Now, we will take this construction to the next level. Let \(T\) be a tangle with
notation $m$ constructed in the manner previously described. First, imagine that there is a diagonal connecting the NW and SE corners of $T$. We must now reflect $T$ about this diagonal (Figure 5). Note that the endpoints of the diagonal remain fixed while the other corners of $T$ are switched. Also, notice that we are not rotating the knot but rather reflecting its projection. Thus, $m$ will maintain its positive or negative orientation after reflection. Next, we can take the right ends of the resulting projection and twist them in a similar manner as before (Figure 6). These new half-twists will be identified in the same manner; suppose that $n$ half twists (left or right handed) were added. Our resulting tangle would have notation $mn$.

![Figure 5: We reflect $T$ about the diagonal.](image)

![Figure 6: The 22 tangle](image)

We could make our tangle more complex by continuing to reflect across the specified diagonal and add half twists. Any tangle that we could construct in this way is called a rational tangle. Also, each rational tangle reflected a $k$ times during its construction has notation $m_1m_2\ldots m_{k+1}$, where $m_1, m_2, \ldots, m_{k+1}$
specify the number and orientation of half-twists added. This notation for knots is due to John H. Conway, and is likewise called Conway notation.

Finally, connecting the north and south ends of $T$ together produces a rational link (Figure 7). Our investigation will be limited to links constructed using rational tangles with Conway notation $mn$, where $m, n \geq 2$. We would now like to discuss two facts about these links that will be useful later.

Figure 7: The link with Conway notation 22, also known as the figure-eight knot.

First, if $m, n$ are both odd, $L$ will have 2 components. Otherwise, $L$ will have only one component (and thus be a knot). To see why this is true, first recall that in constructing $L$ we start with two strands. Let $s_1$ and $s_2$ be these strands. One can easily imagine that if we eventually glue the ends of $s_1$ together, we must also glue the ends of $s_2$ together. In this case, a 2-component link would result. On the other hand, gluing an end of $s_1$ to an end of $s_2$ would force us to glue the other two ends of these strands together. Here, we obtain a knot. Now consider the case where $m$ is even. After twisting $s_1$ and $s_2$ into a tangle of notation $m$ and reflecting that tangle’s projection, the two right ends of the tangle come from $s_2$. Now, the parity of $n$ is arbitrary; after twisting the ends of $s_2$ into a tangle of notation $n$, one end will be a north end of the tangle and the other will be a south end of the tangle. Thus, the ends of $s_2$ are glued to ends of $s_1$ and $L$ is a knot. Figure 8 details this progression. Now, let $m$ be odd. Here, after twisting $s_1$ and $s_2$ into a tangle of notation $m$
and reflecting the tangle, the northwest and southeast ends of the tangle come from $s_1$ and the other two ends come from $s_2$. Thus, we then twist ends from the two different strands into a tangle of notation $n$. If $n$ is even, this twisting preserves the relationship of ends just describes. Thus, we once again glue ends of $s_1$ to ends of $s_2$. However, if $n$ is odd we will obtain a tangle whose northern ends come from $s_1$ and southern ends from $s_2$. Thus, we glue the ends of $s_1$ to each other and likewise for $s_2$, obtaining a link with 2 components. The latter case is laid out in Figure 9.

Second, we claim that every $L$ is alternating; traveling along the link in either direction, we never encounter consecutive over- or under-passes. One can easily see that tangles with single-digit notation are always alternating. Next, note from Figure 6 that when we continue twisting strands after reflection the alternating nature of the tangle is preserved. Specifically, the strand coming from an underpass becomes an overstrand and the strand coming from overpass becomes and understrand. For tangles of notation $mn$, this quality is observed whenever $m$ and $n$ have the same sign. Finally, note that one north end of
When $m$ and $n$ are odd, $L$ has two components. $T$ comes from an overpass and the other comes from an underpass. Thus, connecting these ends preserves the alternating quality. The same can be said for the south ends. Since in our investigation $m$ and $n$ are always positive, the projections we construct using rational tangles will always be alternating.

1.3 The Figure-Eight Knot

Then, the simplest link under investigation is the rational link with Conway notation 22. This link, it turns out, is simply the figure-eight knot. Similarly, the figure-eight knot is also the simplest example of a class of knots called hyperbolic knots. What makes some knots and links hyperbolic will be paramount to our investigation and will be examined shortly. For now, however, we would like to use the figure-eight knot as a visual example to help explain some definitions on the rational links we are investigating.

First, let $\mathcal{L}$ be the collection of links with Conway notation $mn$, $m, n \geq 2$. 
Next, let \( L \in \mathcal{L} \) such that \( L \) has Conway notation \( m_0n_0 \). Since \( L \) has Conway notation, it must be a rational link, i.e. obtainable through the process previously described. Going forward, slightly tweaking the projection of \( L \) as we obtained it from that process will be convenient. Note then that after constructing \( L \) two tangles are salient in its projection: a vertical tangle on the left and a horizontal one on the right (Figure 10a). We would like to rotate this projection of \( L \) a quarter turn counterclockwise such that the left-vertical tangle is now bottom-horizontal and the right-horizontal tangle is now top-vertical (Figure 10b). We refer to any projection so obtained - i.e. by constructing \( L \) and rotating its projection \( 90^\circ \) counterclockwise - as the \textit{standard projection} of \( L \) and will consider only that standard projection whenever dealing with \( L \).

Note in Figure 10 how turning the projection obtained from constructing the 22 rational link yields one of the commonly used projections of the figure-eight knot presented in Figure 2.

Going back to our construction, note that adding \( m \) half-twists to the 0 tangle produces a tangle with \(|m|\) crossings. Also, reflecting the resultant tangle and adding \( n \) more half-twists contributes \(|n|\) more crossings to the projection. Finally, connecting the top and bottom ends of the link adds no crossings. Since for any \( L \in \mathcal{L} \) the numbers \( m \) and \( n \) are both positive, the standard
projection of $L$ will contain exactly $m + n$ crossings. Labeling each crossing will also prove convenient later. Moving down from the top, assign to each consecutive crossing on the top-vertical tangle a label $1, 2, \ldots, a$. Note that since the tangle with notation $n$ was rotated (with the rest of the knot) a quarter turn counterclockwise, the label $a$ represents the total number of crossings in that tangle and is equal to $n$. Next, moving from left to right assign to each consecutive crossing on the bottom-horizontal tangle a label $a+1, a+2, \ldots, a+b$, where $b = m$ is the total number of crossings in that tangle. Figure 11 displays how the crossings of the figure-eight knot should be labeled.

![Figure 11: The figure-eight knot with crossings labeled.](image)

This concludes our introduction to some of the basic ideas of knot theory. Next, we will continue our investigation of rational links in $L$ by examining properties of the space surrounding $L$.

2 Topologically Modeling the Link Complement

2.1 The Link Complement

Let $L \in \mathcal{L}$ and imagine that $L$ lies on a plane (that is, except near the crossings) in $S^3$. For the sake of review, note that $S^3$ is the space obtained by gluing two
solid balls together along their boundaries. We then call the region of space surrounding $L$, $S^3 - L$, the \textit{complement} of $L$. For all $L \in \mathcal{L}$ and for some links in general we can divide the complement into tetrahedra and embed the tetrahedra into $\mathbb{H}^3$, hyperbolic 3-space. The ability to cover $\mathbb{H}^3$ with isometric images of these embedded tetrahedra is what makes some links \textit{hyperbolic}. We will explore a consequence of the hyperbolic quality of these links in Section 3.

2.2 Adams’ Method

In his doctoral thesis dissertation ([1]), Colin Adams presents an easy and useful way to model the complement of any hyperbolic link. Essentially, he divides the complement into a set of tetrahedra; after the faces of these tetrahedra are glued together in a particular manner and their vertices are removed, a link complement is established. We will now present this method in sufficient depth such that it is useful for our investigation. Not only will this allow us to describe later findings in a more concise way, but it also prevents us from considering situations that are not relevant to the subset of links we are investigating.

First, Adams is able to produce edges of the link complement using some clever topological handiwork. That process will be briefly presented here and, in doing so, will be applied to $\mathcal{L}$. Using the method previously described construct a link $L$ with Conway notation $mn$, where $m, n \geq 2$. Then, let $L \in S^3$ and project $L$ onto a plane $P$ so that the projection is identical to that of our construction. Next, near each crossing let $L$ lie on the surface of a crossing sphere $S_i$, for $i = 1, 2, \ldots, a + b$ (Figure 12).

The rest of $L$ will lie in $P$. Next, let $B_i$ be the 3-cell in $S^3 - L$ bounded by $S_i$. Finally, choose coordinates in $R^3 = \{(x_1, x_2, x_3) : x_i \in \mathbb{R}\}$ such that $P = \{(x_1, x_2, x_3) : x_3 = 0\}$. 

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Next, let $C_1$ and $C_2$ denote subsets of $S^3$ such that

$$C_1 = \{(x_1, x_2, x_3) : x_3 \geq 0\} \cup \{\infty\} - L - \bigcup_{i=1}^n \tilde{B}_i$$

and

$$C_2 = \{(x_1, x_2, x_3) : x_3 \leq 0\} \cup \{\infty\} - L - \bigcup_{i=1}^n \tilde{B}_i.$$

Gluing $C_1$ and $C_2$ together via the points they share in $P$ then yields a space homeomorphic to $S^3 - L - \bigcup_{i=1}^n \tilde{B}_i$. The true genius of this method then lies in how we fill in the $B_i$’s. Adams argues that in $S^3$ each $B_i$ is homeomorphic to a regular octahedron $O$ in $R^3$ centered at the origin such that the overstrand lies in the plane corresponding to $x_2 = 0$ and the understrand in $x_1 = 0$ (Figure 13).

Note that two objects that are homeomorphic are related by a homeomorphism: a continuous bijective mapping of one object to the other. Two objects that are homeomorphic are topologically equivalent and are related by the symbol $\cong$.

One could then imagine that since $B_i \cong O$, $B_i - L \cong O - L$. Let $O' = O - L$. We cut $O'$ into four sections, each of which contain two faces from $\partial O'$ and the edge $\{(0,0,t) : -1 \leq t \leq 1\}$. Specifically, sections $C_{12}$ and $C_{34}$ contain the top
Figure 13: Each $B_i$ is homeomorphic to a regular octahedron $O$.

faces of $O'$, where $C_{13}$ and $C_{24}$ contain the bottom faces. Figure 14 shows one of the surfaces Adams uses to divide $O'$.

Figure 14: One of the surfaces dividing $O'$.

Finally, let $C'_1$ and $C'_2$ be the 3-cells obtained by gluing $C_{12}$ and $C_{34}$ to $C_1$ and $C_{13}$ and $C_{24}$ to $C_2$ via the faces they share on $\partial O$. Repeat this procedure for each $B_i$, $i = 1, 2, \ldots, n$.

Then, gluing $C'_1$ and $C'_2$ together via their identifications on $\partial C_1, \partial C_2$ and the $\partial C_{ij}$'s from each $B_i$ produces a structure homeomorphic to $S^3 - L$. Also, note that since $C_{12}$ and $C_{34}$ share a vertical edge, two arcs from $\partial C'_1$ will be identified to each other for each crossing in $L$, $i = 1, 2, \ldots, a+b$. The same is true for $C'_2$. Adams then suggests that “a good way to picture this construction is to think of pinching the top 3-cell around the overstrand in an edge running
from the overstrand to the understrand and pinching the bottom 3-cell around the understrand so that it touches itself in that same edge”. A visualization from Adams’ thesis is provided in Figure 15.

![Figure 15: Identifying two boundary edges by pinching around the overstrand.](image)

These edges lie on the boundary of the complement of $L$, are oriented upward and will be the edges along which we eventually glue $C'_1$ to $C'_2$.

Up to this point, Adams’ method essentially explains why the steps that follow will produce a structure homeomorphic to the complement of $L$. Thus, the following procedure describes exactly how that structure is formed. In order to demonstrate the method visually we will need a simple, concrete example that requires us to utilize the method in its entirety. Thus, let $L_0$ be the link with Conway notation $3\overline{3}$, as shown in Figure 16.

![Figure 16: $L_0$, the link with Conway notation $3\overline{3}$](image)
Fortunately, we can begin to represent the spatial relationship of the edges on $\partial C'_1$ and $\partial C'_2$ using the projection of $L$. Once again, let the projection of $L$ appear as we constructed it. Then, upon flattening $\partial C'_1$ into a plane, the corresponding edges of $\partial C'_1$ will appear as pairs of segments of each understrand separated by a single point (Figure 17a). Note that since the single vertical edge from Figure 15 is oriented upward, these edges are oriented toward their respective crossings. Also, at each crossing let both segments be labeled with the number associated with that crossing; this number denotes the “type” of those edges, and designates which edges will be identified when our construction is complete. Similarly, the corresponding edges of $\partial C'_2$ appear as pairs of segments of the overstrand oriented away from the crossing and separated by a single point and are labeled in the same way (Figure 17b). Finally, note that since a pair of segments lie near each crossing both $\partial C'_1$ and $\partial C'_2$ contain $2(m + n)$ such edges.

Figure 17: The boundary edges of $C'_1$ and $C'_2$ with appropriate orientations and labels.

Next, let $A_1$ and $A_2$ be the projections of $L$ representing $\partial C'_1$ and $\partial C'_2$, respectively. Note that there remain arcs on $A_1$ and $A_2$ that are neither labeled nor oriented; these are the arcs of $L$ that we did not include in the boundaries of the $C_{ij}$’s. We can now split along the missing arcs and shrink the resulting open sets down to missing points. Specifically, the arcs will be shrunk down to the points separating the edges of a given type at each crossing, as in Figure 18.
Figure 18: The missing arcs are shrunken down to missing points.

Next, we would like to fill in these missing points so that $A_1$ and $A_2$ become connected planar graphs; the points will become vertices of tetrahedra that will be removed when the tetrahedra are embedded in $\mathbb{H}^3$. Also, note that since a point is added at each crossing our planar graphs contain as many vertices as the projection of $L$ does crossings, $m + n$. Finally, let $C''_1$ and $C''_2$ be the 3-cells with the missing points replaced and let $A'_1$ and $A'_2$ be the connected planar graphs lying on their boundaries.

Now, note that $A'_1$ and $A'_2$ divide $\partial C''_1$ and $\partial C''_2$ into regions that we will call faces. For a link $L \in \mathcal{L}$ with Conway notation $mn$, each graph will divide the boundary of its corresponding 3-cell into $m + n + 2$ such faces. A simple proof of this fact is to note that the boundary of a 3-cell is topologically equivalent to a sphere, which has Euler characteristic 2. The Euler characteristic of a surface is equal to the sum of the number of vertices and faces minus the number of edges on that surface, or $\chi = V - E + F$. We have already shown that for $A'_1$ and $A'_2$, $V = m + n$ and $E = 2(m + n)$. Then, since these graphs lie on the boundaries of 3-cell, $2 = (m + n) - 2(m + n) + F$ implies $F = m + n + 2$ (Figure 19).
Figure 19: For $L_0$, $\partial C''_1$ and $\partial C''_2$ have $3 + 3 + 2 = 8$ faces each.

2.3 Collapsing the Digons

Now, note that some of the faces on $\partial C''_1$ and $\partial C''_2$ are bound by only two edges. Adams calls such faces “digons”, and one can easily see that they come from the component tangles of $L$. Specifically, from a tangle of Conway notation $m$ there will result $m - 1$ digons (Figure 20).

Figure 20: A tangle with notation 3 corresponds to 2 digons.

Thus, of the $m + n + 2$ faces on $\partial C''_1$, $m + n - 2$ are digons. Let $F_1$ be a digon on $\partial C''_1$ bound by edges $e_1$ and $e_2$ with types $t_1$ and $t_2$, respectively. Note that $t_1 \neq t_2$. Also, let $F_2$ be the digon on $\partial C''_2$ to which $F_1$ identifies. Adams provides a mapping that identifies all the points on $F_1$ to points on $e_1$. Consequently, all points on $F_2$ identify to the edge $e'_1$ to which $e_1$ identifies.

Now that $e_2$ is identified to $e_1$, the edge type $t_2$ becomes irrelevant. Thus, the total number of edge types decreases by 1. Since Adams dictates that we perform this process (which we will hereafter refer to as a “digon collapse”) for each digon $d$ on $\partial C''_1$, the total number of edge types reduces to 2. To see why
this happens, recall that before shrinking down the missing arcs of $C'_1$ and $C'_2$ we assigned to each edge a type equal to the label of its corresponding vertex. Upon shrinking down the missing arcs, these edges were extended out from their respective crossings and toward adjacent crossings. Further, we labeled these adjacent crossings with consecutive integers. Thus, each digon will have an edge with type $t$ and another edge with type $t + 1$. Therefore, after we collapse all digons corresponding to the top-vertical tangle of $L$, edges with consecutive types 1 through $a$ become identified, i.e. 1 to 2, 2 to 3, up to $a$. We assign the single type 1 to all edges that result from collapsing digons correspondent to the top-vertical tangle. Also, since a digon with an edge of type $a$ is collapsed during this process, the remaining edge with type $a$ will also be reassigned the type 1. In the bottom-horizontal tangle, edges with consecutive types $a + 1$ through $a + b$ are identified and we use 2 to describe the type of edges that result from digon collapse there. Furthermore, the remaining edges with type $a + 1$ and $a + b$ will also acquire the new type 2. Figure 21 shows the boundary of $C''_1$ after we collapse all of its digons.

![Figure 21: Collapsing all of the digons on $\partial C''_1$.](image)

Next, after digon collapse we must adjust the orientation of resulting edges. Note from Figure 21 that the component edges of each digon have opposite orientations. Since upon collapsing each digon we discard an edge type, we
must preserve the orientation of the edge type that remains. Specifically, note that before digon collapse the edge with type 1 has orientation opposite that of the edge with type 2, whose orientation is opposite that of the edge with type 3. Thus, we could then say that the edges with type 1 and 3 have equivalent orientation. Generally, edges with odd type up to \(a\) will have equivalent orientation. When collapsing a digon \(d_1\) correspondent to the top-vertical tangle we assign the resulting edge orientation equivalent to that of the odd-type edge of \(d_1\). Also, note that if \(a\) is even we must reverse the orientation of the “remaining” edge with type \(a\) previously mentioned. Revert to Figure 21 to verify these claims.

Upon collapsing digons correspondent to the bottom-horizontal tangle, assigning orientations is slightly more complicated. The lowest type of any edge on these digons is equal \(a + 1\). Thus, after collapsing a digon \(d_2\) on this tangle we assign the resulting edge orientation equivalent that of the edge of \(d_2\) whose parity is the same as \(a + 1\). Also, if the parity of \(a + b\) does not match that of \(a + 1\), we must reverse the orientation of the “remaining” edge with type \(a + b\).

Finally, note the following overall effects of collapsing the digons on \(\partial C''_1\). First, all edges of the remaining structure have one of two types. Second, if two edges of the same type share a vertex, both of these edges are oriented either toward or away from that vertex. Third, the general structure of \(L\) guarantees that collapsing the original digons will not create new ones. Before digon collapse, \(\partial C''_1\) contains \(2(m+n)\) edges and \(m+n-2\) digons. Thus, there are \(2(m+n)-2(m+n-2) = 4\) edges of \(\partial C''_1\) that do not lie on a digon and essentially connect each end of the digon ”chains” corresponding to the component tangles of \(L\). Since before digon collapse \(\partial C''_1\) contains \(m + n + 2\) faces: Fourth, after collapsing each digon there remain \((m + n + 2) - (m + n - 2) = 4\) faces on \(\partial C''_1\), each of which are bound by three or more edges (Figure 22). We observe the
same effects after collapsing the digons of $\partial C''_2$.

![Figure 22: After collapsing the digons, $C''_1$ and $C''_2$ have 4 faces each.](image)

It is then clear that since $C''_1$ and $C''_2$ are 3-cells we are left with two polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$. Therefore, when the corresponding faces of $\mathcal{P}_1$ and $\mathcal{P}_2$ are identified and their vertices are removed we will be left with a space homeomorphic to $S^3 - L$.

### 2.4 Slicing $\mathcal{P}_1$ and $\mathcal{P}_2$ into Tetrahedra

We would now like to slice $\mathcal{P}_1$ and $\mathcal{P}_2$ into tetrahedra. To accomplish this, we must triangulate every face of the two polyhedra (a point that we will elaborate on shortly). Since each face on $\mathcal{P}_1$ identifies with a face on $\mathcal{P}_2$, we will first triangulate $\partial \mathcal{P}_1$. We will do this by choosing a vertex $v_i$ on each face $F_i$ of $\mathcal{P}_1$ and drawing an edge from $v_i$ to all other vertices on $F_i$ with which $v_i$ does not already share an edge. Figure 23 demonstrates why this “fanning out” technique always triangulates a given face.

![Figure 23: “Fanning out” edges from a vertex $v_i$ always produces an acceptable triangulation.](image)

Then, for each face on $\mathcal{P}_1$ we will add edges to $\partial \mathcal{P}_2$ so that the faces of $\mathcal{P}_2$
match up with those of $P_1$. In order to simplify this process, we will continue to utilize the planar graphs. Thus, let $A_1''$ and $A_2''$ be the planar graphs that result from collapsing the digons. Also, label the vertices of $A_1''$ and $A_2''$ according to the crossings in the projection of $L$ to which they correspond. For the sake of clarity, vertex labels are circled in each figure, as in Figure 24. Also, label each face of $A_1''$ and $A_2''$ as in Figure 24. Note that the label “D” corresponds to the face of $P_1$ whose component edges are the perimeter of $A_1''$. Also, note that since $a = n$ and $b = m$, faces $A$ and $B$ contain $n + 1$ vertices and faces $C$ and $D$ contain $m + 1$ vertices. Additionally, since these faces are polygons they will each contain as many edges as they have vertices.

![Figure 24: $A_1''$ with vertices, edges and faces labelled.](image)

Table 2.4a shows for each face $F_i$ the vertex $v_i$ from which we choose to fan out edges. The opposite endpoints of these edges are denoted $v_{ij}$. Also, the fourth column of Table 2.4 describes the formulas for determining the type $t_k$ of an edge $e_k$ with endpoints $v_i$ and $v_{ij}$. Essentially, these formulas guarantee that we assign as many consecutive odd-valued types as necessary to faces $A$ (first) and $C$ (second) and as many even-valued types as necessary to faces $B$ (first) and $D$ (second).
Additionally, each edge added to $\partial P_1$ will be oriented toward its corresponding $v_i$. Finally, if a face $F_i$ of $P_1$ is already a triangle (as when $a = 2$), we need not add edges to it. Note that for any $L \in \mathcal{L}$ and its corresponding standard projection, our vertex labeling system dictates that vertex 1 will always be the uppermost vertex. Also, vertices $a$, $a + 1$ and $a + b$ will always be the center, leftmost and rightmost vertices, respectively. Figures 25 through 28 show the new edges added to $P_1$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$F_i$ & $v_i$ & $v_{ij}$ & $t_{ij}$ \\
\hline
$A$ & 1 & $j_1 \in \{3, 4, \ldots, a\}$ & $2j_1 - 3$ \\
\hline
$B$ & $a + b$ & $j_2 \in \{2, 3, \ldots, a - 1\}$ & $2j_2$ \\
\hline
$C$ & $a$ & $j_3 \in \{a + 2, a + 3, \ldots, a + b - 1\}$ & $2j_3 - 5$ \\
\hline
$D$ & $a + b$ & $j_4 \in \{a + 1, a + 2, \ldots, a + b - 2\}$ & $2j_4 - 2$ \\
\hline
\end{tabular}
\caption{Types of Edges Added to $\partial P_1$}
\end{table}

Figure 25: We add an edge of type 3 to face $A$.

Figure 26: We add an edge of type 4 to face $B$.

Also, note that triangulating $\partial P_2$ produces $2(m + n - 4)$ new edges, each
with distinct type. We mentioned previously that face $A$ is a polygon with $m+1$ vertices. A given vertex $v_i$ on $A$ thus shares an edge with 2 adjacent vertices, and we do not add an edge from $v_i$ to itself. Thus, in triangulating $A$ we add $(n + 1) - 3 = n - 2$ edges with distinct type. The same holds for face $B$, and one can easily determine that we therefore add $m - 2$ new edges to each of faces $C$ and $D$. Since $\partial P_1$ contained only two types of edges before triangulation, it contains $2(m - 2) + 2(n - 2) + 2 = 2(m + n - 4) + 2 = 2(m + n - 3)$ edge types following triangulation. Since each face on $\partial P_1$ identifies to a face on $\partial P_2$, the latter surface will also contain $2(m + n - 3)$ different types of edges (although these will not be distinct from their counterparts on $\partial P_1$).

Now that we have shown how to triangulate the faces of $P_1$, we must explain why this process is sufficient to slice $P_1$ into tetrahedra. First, imagine that we have four vertices such that they do not all lie in the same plane. Now, connect the vertices such that each pair of vertices are the endpoints of exactly one line segment. Our resulting structure then has $3! = 6$ segments and is a tetrahedron.
Now, consider how we triangulate faces $A$ and $B$. For each triangle $\Delta_B$ on $B$ that does not contain vertex 1 there is a triangle $\Delta_A$ on $A$ that shares an edge with $\Delta_B$. Thus, $\Delta_A$ and $\Delta_B$ share two vertices; note from Figure 29 that neither of the vertices are 1 nor $a + b$. In fact, 1 and $a + b$ are always the vertices on $\Delta_A$ and $\Delta_B$, respectively, that those triangles do not share.

![Figure 29: Triangulating faces $A$ and $B$ of $P_1$ produces a group of tetrahedra glued together along faces in the interior of $P_1$.](image)

Finally, even before triangulation $\partial P_1$ contained an edge with 1 and $a + b$ as endpoints. Thus, $\Delta_A$ and $\Delta_B$ are the faces of an object $T_1$ containing four vertices (the two vertices from their shared edge, 1 and $a + 1$) such that each pair of vertices are the endpoints of exactly one segment. Note that since we are working in a planar graph all of the aforementioned vertices lie in the same plane. We now dictate that upon triangulating the faces of $P_1$ vertices 1 and $a + 1$ through $a + b$ will lie on a plane $P_0$ and vertices 2 through $a$ will lie on a parallel plane $P_0'$. Also, since we originally viewed the edges of $\partial P_1$ from the north pole of $S^3$ we will imagine $P_0'$ as lying below $P_0$. Thus, all four vertices of $T_1$ do not lie in the same plane. Therefore, $T_1$ must be a tetrahedron. Note that the right side of Figure 29 depicts a group of tetrahedra with faces glued together in the interior of $P_1$. These faces can be imagined as lying on planes that cut through at different angles the segment with endpoints at 1 and $a + b$. 

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Also note that while one triangle on $A$ is not a face of any tetrahedra (right side of 29), the triangles with vertices at 1, 2, $a + 1$ and 1, $a, a + 1$ are the faces of "outer" tetrahedra that are not yet shared. Figure 30 provides a simpler (and perhaps more convincing) visualization.

![Figure 30: Triangulating faces $A$ and $B$ in this simple case produces a single tetrahedron.](image)

Next, triangulating faces $C$ and $D$ will complete the process of slicing $\mathcal{P}_1$ into tetrahedra. Like before, triangulating these faces will produce a finite set of tetrahedra. Furthermore, adding an edge from $a + 1$ to $a + b$ will serve to incorporate the triangle on $A$ that had not yet become a face on a tetrahedron. Note first that each triangle $\Delta_C$ on $C$ that does not contain vertex $a + b$ shares an edge with a triangle $\Delta_D$ on $D$. The vertices that $\Delta_C$ and $\Delta_D$ do not share are $a$ and $a + b$. Furthermore, before triangulation $\partial \mathcal{P}_1$ contained an edge with endpoints at $a$ and $a + 1$. Thus, $\Delta_C$ and $\Delta_D$ are faces of a structure $\mathcal{T}_2$ containing four vertices where each pair of vertices are the endpoints of exactly one line segment. Since these four vertices do not lie in the same plane, $\mathcal{T}_2$ is a tetrahedron. Figure 31 displays these qualities in a general fashion, while Figure 32 shows a simpler case. Next, Figure 33 demonstrates that upon completely triangulating $\mathcal{P}_1$ a tetrahedron is created with vertices at 1, $a, a + 1$ and $a + b$ that incorporates the previously loose face of $A$ and is incident to neither tangle of $L$. Finally, Figure 34 presents the full triangulation of $\partial \mathcal{P}_1$ in the 33 rational link complement.
Figure 31: Triangulating faces $C$ and $D$ of $P_1$ produces a group of tetrahedra glued together along faces in the interior of $P_1$.

Figure 32: Triangulating faces $C$ and $D$ in this simple case produces a single tetrahedron.

Figure 33: Adding an edge with endpoints $a + 1$ and $a + b$ completes the process of slicing $P_1$ into tetrahedra.
Now that we have shown why triangulating the faces of $\partial \mathcal{P}_1$ effectively slices $\mathcal{P}_1$ into tetrahedra, we will turn to the issue of triangulating $\partial \mathcal{P}_2$. Note that each face of $\mathcal{P}_1$ identifies to a face on $\mathcal{P}_2$. Thus, we will use the same face labels (Figure 35) for $\mathcal{P}_2$ because the two polyhedra share each face.

Fortunately, we can triangulate $\partial \mathcal{P}_2$ in a manner nearly identical to that used for $\partial \mathcal{P}_1$, with only a few differences. First, since each face of $\mathcal{P}_1$ identifies to a face of $\mathcal{P}_2$, the choices of $v_i$ we made previously dictate the vertices from which we must fan out edges on $\partial \mathcal{P}_2$. Second, each new edge is oriented toward from $v_i$. Third, vertices 1 and $a+2$ through $a+b−1$ will lie on a plane $P_1$ while vertices 2 through $a$, $a+1$ and $a+b$ will lie on a parallel plane $P'_1$ such that $P'_1$ appears to lie above $P_1$. If we thus imagine that the surface of this page is
$P_1$, the edges fanning out from $a + 1$ will appear to stick out of the page. Table 2.4b shows the edges we add to the faces of $P_2$ and formulas for determining their types.

<table>
<thead>
<tr>
<th>$F_i$</th>
<th>$v_i$</th>
<th>$v_{ij}$</th>
<th>$t_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$a + 1$</td>
<td>$j_2 \in {2, 3, \ldots, a - 1}$</td>
<td>$2j_2 - 1$</td>
</tr>
<tr>
<td>$B$</td>
<td>1</td>
<td>$j_1 \in {3, 4, \ldots, a}$</td>
<td>$2j_1 - 2$</td>
</tr>
<tr>
<td>$C$</td>
<td>$a + b$</td>
<td>$j_4 \in {a + 1, a + 2, \ldots, a + b - 2}$</td>
<td>$2j_4 - 3$</td>
</tr>
<tr>
<td>$D$</td>
<td>1</td>
<td>$j_3 \in {a + 2, a + 3, \ldots, a + b - 1}$</td>
<td>$2j_3 - 4$</td>
</tr>
</tbody>
</table>

As the demonstrations would be very similar, we will not go over why triangulating $\partial P_2$ in this manner always slices $P_2$ into tetrahedra. Figure 36 provides the complete triangulation of $\partial P_2$ for the 33 rational link complement.

Upon successfully slicing $P_1$ and $P_2$ into tetrahedra we have successfully modeled the compliment of $L$. Now, we claim that this model contains the same number of tetrahedra that it does edge types, $2(m + n - 3)$. To demonstrate this, we first consider the triangulation of face $A$ on $\partial P_1$. Note that after adding the first new edge to face $A$, a triangle not containing vertex 1 is created as we draw additional edges. Furthermore, adding the edge that completes the triangulation of $A$ creates two triangles not containing vertex 1. Thus, triangulating $A$ creates
\( n - 2 \) such triangles, the same as the number of new edges added (0 triangles for the first edge, 2 for the last and 1 for each edge in between).

![Diagram](image.png)

Figure 37: Triangulating \( A \) splits it into \( n - 1 \) triangles, only one of which contains the vertex 1.

Now, recall that upon triangulating faces \( A \) and \( B \) a tetrahedron is created for each triangle \( \Delta_A \) on \( A \) that does not contain the vertex 1. Therefore, triangulating these faces creates \( n - 2 \) tetrahedra. Similar arguments would show that triangulating faces \( A \) and \( B \) on \( \partial P_2 \) also creates \( n - 2 \) tetrahedra and that triangulating faces \( C \) and \( D \) on \( \partial P_1 \) and \( \partial P_2 \) creates \( m - 2 \) tetrahedra each. Finally, recall from Figure 33 that completely triangulating \( \partial P_1 \) creates a tetrahedron not incident to either tangle. The same effect occurs on \( \partial P_2 \). Thus, our model of \( S^3 - L \) contains \( 2(m - 2) + 2(n - 2) + 2 = 2(m + n - 3) \) tetrahedra.

In the next section, we show that these tetrahedron are organized in pairs such that for each tetrahedron in \( P_1 \) there is an identically equal tetrahedron in \( P_2 \).

### 3 Finding the Hyperbolic Structure

#### 3.1 Hyperbolic Links

We would now like to embed \( P_1 \) and \( P_2 \) into \( \mathbb{H}^3 \) in order to determine the hyperbolic structure of \( L \). In doing so, we first glue \( P_1 \) to \( P_2 \) via one of the faces.
they share, creating a single polyhedron $\mathcal{P}$. We then embed $\mathcal{P}$ into $\mathbb{H}^3 \cup S^2_\infty$ with all vertices lying on $S^2_\infty$ and all faces lying in geodesic planes. Upon embedding $\mathcal{P}$, we want to realize its facial identifications as hyperbolic isometries such that these isometries generate a discrete group $\Gamma$ (i.e. “tile” $\mathbb{H}^3$). $\mathcal{P}$ will then act as a fundamental polyhedron for $\Gamma$ and $S^3 - L \cong \mathbb{H}^3 / \Gamma$ will admit a hyperbolic structure. We then say that $L$ is hyperbolic. To determine the hyperbolic structure of $S^3 - L$, we develop, simplify and solve a system of $2(m + n - 3)$ equations in the same number of variables, each of which correspond to an individual tetrahedron in $\mathcal{P}$. Finally, we determine the hyperbolic volume of $S^3 - L$, equal to the total finite volume of a single image of $\mathcal{P}$ in $\mathbb{H}^3$.

3.2 Embedding the Link Complement

Now, note that we embed the component tetrahedra of $\mathcal{P}$ as ideal tetrahedra with vertices on $S^2_\infty$ and faces on geodesic planes. The intersection angles of these planes are known as dihedral angles, and are essential to the constructions in this section. As such, consider the following properties of ideal tetrahedra (as presented by Adams):

1. The sum of the dihedral angles about any one ideal vertex is $\pi$.
2. Opposite dihedral angles are equal.
3. The dihedral angles about any one ideal vertex determine the tetrahedron up to isometry.

Let $\mathcal{T}$ be a tetrahedron in the upper half-space model of $\mathbb{H}^3$ and put one of the vertices of $\mathcal{T}$ at $\infty$. Then, Figure 38 (taken from [4]) is helpful in visualizing these properties. First, a horosphere cross-section of $\mathcal{T}$ produces a Euclidean triangle that displays the dihedral angles of $\mathcal{T}$. A horosphere is a ball in $\mathbb{H}^3$ that intersects the boundary at exactly one point; in the upper half-space model,
horospheres that intersect the boundary at \( \infty \) appear as horizontal planes. Since
the intersection of this plane and \( T \) produces a Euclidean triangle with angles
equal to the dihedral angles of \( T \), those angles must have sum \( \pi \). Next, Adams
draws a geodesic edge perpendicular to two opposite edges of \( T \). Rotating \( T \)
180° about this geodesic takes those opposite edges back to themselves but
interchanges their respective vertices. Also, this action sends the other pairs
of opposite edges to each other. As such, the vertices of \( T \) are permuted and
\( T \) is taken back to itself. Thus, the two other pairs of opposite edges must
have the same dihedral angles. It then follows that the pair intersecting the
godesic have equal dihedral angles as well. Finally, the invariant geometric
properties (including hyperbolic volume) of ideal tetrahedra can be determined
as a function their dihedral angles. Since the isometries of \( \mathbb{H}^3 \) (the Mobius
transformations and \( T(z) = \bar{z} \)) take tetrahedra with the same set of dihedral
angles to each other, the dihedral angles are said to determine the tetrahedra
up to isometry.

![Figure 38](image)

Figure 38: The dihedral angles of an ideal tetrahedron have properties essential
to our construction.

Thus, all possible isometry classes of ideal tetrahedra can be determined
using similarity class of Euclidean triangles. Adams then presents a method for
parametrizing this similarity class: put a Euclidean triangle \( \Delta \) in the complex
plane such that two of its vertices lie at 0 and 1 (respectively) and the third
lies at a point $z$ with positive imaginary part. We could then easily determine the angles of $\Delta$ and thus the angles of the ideal tetrahedron it represents. Also, note that if $z_1 = z$ corresponds to placing the vertex $v_1$ of $\Delta$ at the origin, then $z_2 = \frac{z - 1}{z}$ and $z_3 = \frac{1}{1 - z}$ correspond to placing the other two vertices at 0, moving clockwise around $\Delta$.

Therefore, $U = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$ modulo the equivalence class $z \sim w$ if $z = w_1, w_2, \text{or} \, w_3$ parametrizes all possible isometry classes of ideal tetrahedra. Thus, we assign complex variables $z^i, i = 1, 2, \ldots, 2(m + n - 3)$ to the component tetrahedra $T_i$ of $\mathcal{P}$. These will be the variables in our system of $2(m + n - 3)$ equations. Note that a particular solution to this system determines the embedding of $\mathcal{P}$ up to a hyperbolic isometry and determines the face identifying isometries up to conjugation by that same hyperbolic isometry.

For our investigation, we would like to assign each $z^i$ in a very specific way. Also, since a given $z^i$ is equivalent to three complex constants that correspond to the dihedral angles of $T_i$, we need to assign $z_1^i, z_2^i$ or $z_3^i$ to the corresponding edges of $T_i$. Note that since opposite edges in $T_i$ have equal dihedral angles, they will also share an edge variable $z_j^i$. Before specifying how the constants are assigned, we want to introduce a specific manner of orientating the tetrahedra. This will allow us to show very clearly how we assign constants to the tetrahedra and their edges and, eventually, how the overall structure of $\mathcal{P}$ changes as twists are added to the tangles of $L$.

Consider once again the disjoint polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$. Note from our triangulations of the boundaries of these polyhedra that every tetrahedron in $\mathcal{P}_1$ contains vertex $a + b$ and every tetrahedron in $\mathcal{P}_2$ contains vertex 1. We want to orient $\mathcal{P}_1$ such that all of its vertices except $a + b$ lie in the same plane and $a + b$ lies above this plane. Next, reorient $\mathcal{P}_2$ such that 1 lies above a plane containing the remaining vertices of $\mathcal{P}_2$. Figure 39 depicts these transformations.
Figure 39: Spatial orientations of $\mathcal{P}_1$ and $\mathcal{P}_2$ that will be useful going forward.

Now, recall that completing our triangulations of $\partial \mathcal{P}_1$ and $\partial \mathcal{P}_2$ creates tetrahedra with vertices at $1$, $a$, $a + 1$ and $a + b$ in the respective polyhedra. In $\mathcal{P}_1$, we call this tetrahedron $T_1$ and assign it the variable $z^1$. In $\mathcal{P}_2$, we assign $T_2$ the variable $z^2$. For all other $T_i$ in $\mathcal{P}_1$, let $i_1$ be the maximum odd type of the edges of $T_i$. Similarly, for all other $T_{i_2}$ in $\mathcal{P}_2$ let $i_2$ be the maximum even type of the edges of $T_{i_2}$. Then, these tetrahedra will be assigned corresponding versions of variables $z^{i_1}$ and $z^{i_2}$. Figure 40 displays the variable assigned to each tetrahedron on $\mathcal{P}_1$ and $\mathcal{P}_2$.

Figure 40: The tetrahedra of $\mathcal{P}$ with variables assigned.
This “function” for assigning variables is one-to-one in that each $T_{i_1}$ has a unique maximum odd edge type and each $T_{i_2}$ has a unique maximum even edge type. A proof here would be very tedious and involve considering several cases. Nevertheless, we provide images of $P_1$ corresponding to the links with Conway notation 34 and 43; the reader can verify that our rules act soundly on these examples.

![Diagram of tetrahedra](image)

Figure 41: Each tetrahedron $T_{i_1}, i_1 \neq 1$ in $P_1$ has a unique maximum edge type.

Now, for each pair of opposite edges on $T_i$ we must assign one of $z_{i_1}^1, z_{i_2}^1$ and $z_{i_3}^1$. First, recall that if $z_1 = z$ corresponds to placing a vertex $v_1$ of a Euclidean triangle $\Delta$ at the origin, $z_2 = \frac{z-1}{z}$ and $z_3 = \frac{1}{1-z}$ correspond to placing the other vertices of $\Delta$ at the origin moving clockwise around $\Delta$. Consider an ideal vertex $v$ of $T_i$. If we assign $z_{i_1}^1$ to an edge at $v$ it then follows that we must assign $z_{i_2}^1$ to the next edge we meet going clockwise around $v$, and so on for $z_{i_3}^1$. Thus, assigning a variable to a single edge on $T_i$ will determine which variables we must assign to the remaining edges of $T_i$. Therefore, our assignment rules here need only to dictate a variable $z_{i_j}^1$ for one edge of each $T_i$.

We will now dictate the variable assignments to be used in our investigation. Let $e = (v_1, v_2)$ denote an edge with vertices $v_1, v_2$. Note from Figures 29 and 31 that two groups of tetrahedra on $P_1$ share the edges $(1, a+b)$ and $(a, a+b)$,
respectively. Similarly, two groups of tetrahedra on $P_2$ share the edges $(1, a+1)$ and $(1, a+b)$, respectively. These are the edges to which we will assign variables. This is convenient because not only will we have few assignment rules, but we will also find that this system for assigning variables will allow us to easily simplify our $2(m + n - 3)$ equations. Table 3.2 specifies the tetrahedra that share the aforementioned edges and the variables assigned to those edges and 42 provides a visualization. Note that in the latter figure the diamonds surrounding $z_i^j$ indicate that we use this variable when considering $T_i$ or $T_j$.

Table 3.2: Edge Variable Assignment Rules

<table>
<thead>
<tr>
<th>$T_i$</th>
<th>$e_j$</th>
<th>$z_i^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1, T_3, \ldots, T_{2n-3}$</td>
<td>$(1, a + b)$</td>
<td>$z_1^i$</td>
</tr>
<tr>
<td>$T_{2n-1}, T_{2n+1}, \ldots, T_{2(m+n-3)-1}$</td>
<td>$(a, a + b)$</td>
<td>$z_4^i$</td>
</tr>
<tr>
<td>$T_2, T_4, \ldots, T_{2n-2}$</td>
<td>$(1, a + 1)$</td>
<td>$z_1^i$</td>
</tr>
<tr>
<td>$T_{2n}, T_{2n+2}, \ldots, T_{2(m+n-3)}$</td>
<td>$(1, a + b)$</td>
<td>$z_3^i$</td>
</tr>
</tbody>
</table>

Figure 42: The tetrahedra of $P$ with edge variables assigned.
3.3 Satisfying the Edge Conditions

Now that we have assigned a variable to each edge of $\mathcal{P}$, we will develop our system of $2(m + n - 3)$ equations by applying Poincare’s theorem for fundamental polyhedra of discrete groups acting on $\mathbb{H}^3$. By this theorem, $z^1, z^2, \ldots, z^{2(m+n-3)}$ must satisfy several conditions. First, after identifications have been made and $\mathcal{P}$ is embedded the sum of angles about a given edge must equal $2\pi$. Second, if we form a product of face identifying isometries that preserve an edge, that product is the identity. We call these the edge conditions; Adams provides a framework for ensuring that each edge of $\mathcal{P}$ satisfies them.

Let $e_1$ be an edge of given type in $\mathcal{P}$ and let $T_1, T_2, \ldots, T_k$ be the tetrahedra involving $e_1$. After embedding $\mathcal{P}$, we can put one of the vertices of $e_1$ at $\infty$ such that $T_1, T_2, \ldots, T_k$ project to the bounding plane as in Figure 43 (taken from Adams).

Figure 43: We project tetrahedra involving $e_1$ to the bounding plane of $\mathbb{H}^3$.

Adams then argues that $z^1z^2z^3z^4z^5 = 1$ assures that the product of isometries preserving $e_1$ is the identity and that the sum of angles around $e_1$ is $2\pi M_1$ for some integer $M_1$. To understand why this is true, consider the polar form $z = |z|e^{\text{Arg} z}$. Then, $z^1z^2z^3z^4z^5 = 1 = 1e^{\text{Arg} 1}$ implies $\prod_{i=1}^5 |z^i| = 1$ and

36
\[
\left[ \sum_{i=1}^{5} \text{Arg } z^i \right] = \text{Arg } 1 = 2\pi M.
\]

Here, \( \prod_{i=1}^{5} |z| = 1 \) guarantees that we have scaled the triangles in Figure 43 so that any two edges (representing faces in \( \mathbb{H}^3 \)) that identify have the same length. Recall that assigning \( z^i \) to a tetrahedron corresponds to placing the vertices of a Euclidean triangle (whose angles represent the dihedral angles of the tetrahedron) at 0, 1 and \( z^i \) in the complex plane. If we place the triangle corresponding to \( z^1 \) in the plane in precisely this way, note in Figure 44 how we must scale the other triangles so that their edges fit together.

![Figure 44: Dictating that \( z^1 z^2 z^3 z^4 z^5 = 1 \) guarantees that we can scale these triangles to fit nicely around \( e_1 \).](image)

Then, we see that \( \prod_{i=1}^{5} |z| = 1 \) guarantees a nice picture. Since this grouping of triangles is really the projection of tetrahedra into the bounding plane of \( \mathbb{H}^3 \), the scales we apply to each triangle correspond directly to the aforementioned face-identifying isometries. Applying these scales takes the edges in Figure 44 back to themselves, signifying that the product of face-identifying isometries preserving \( e_1 \) is the identity, taking the faces that involve \( e_1 \) back to themselves.

Next, Adams presents and proves a lemma claiming that if we have \( N \) edge types, we need only satisfy \( N - l \) edge equations, where \( l \) is the number of cusps.
of $S^3 - L$. A cusp is equivalent to a tubular neighborhood of $L$; essentially, it is the subset of $S^3 - L$ that is closest to (but does not touch) $L$. The number of cusps is thus equal to the number of components of $L$. Therefore, in our investigation $l$ will equal either 1 or 2.

To present Adams’ proof, we first need to construct what he calls a link diagram. To do so, pick a vertex $x_0$ on any tetrahedron $T_i$ in $\mathcal{P}$ and put $x_0$ at $\infty$ in the upper half-space model. Now, the projection of $T_i$ to the bounding plane is a Euclidean triangle. Next, to each face on $T_i$ with a vertex at $x_0$ glue the appropriate tetrahedra via the face-pairing isometries. Then, continue to glue tetrahedra to each face added to our structure with a vertex at $\infty$. However, do not glue tetrahedra to faces whose counterparts are already in our structure and have a vertex at infinity. Thus, this process ends exactly when each unglued face containing a vertex at $\infty$ has the quality just described. This polyhedron we have constructed projects to a polygon in the bounding plane: this polygon is a link diagram.

We now argue that each link diagram corresponds to a cusp of $S^3 - L$. Recall from Section 1 that the we shrank the arcs of $L$ missing from $C''_1$ and $C''_2$ down to points, then filled in the points. These are the same points (in the form of ideal vertices) that we remove when we embed $\mathcal{P}$ into $\mathbb{H}^3$. These actions signify that if the complement of $L$ tiles $\mathbb{H}^3$, $L$ can essentially be found on the boundary of $\mathbb{H}^3$. Thus, in hyperbolic space the cusps are realized as the sharp corners of our embedded polyhedron (minus the ideal vertices, of course) since they lie close to the boundary. We can then take a cross-section of the cusp using a horosphere whose point of intersection with the boundary is exactly the ideal vertex corresponding to the cusp. If that particular ideal vertex is located at $\infty$, the horosphere will appear as a plane horizontal to the bounding plane in the upper half-space model. Thus, taking the cross-section
of tetrahedra will produce an image equivalent to that of projecting them to the bounding plane. Therefore, we see that link diagrams are neighborhoods of horosphere cross-sections of the cusp; each link diagram therefore corresponds to a cusp of $S^3 - L$.

Figure 45 displays one of the link diagrams for $L_0$. In this figure, one edge variable within each triangle is specified, as are the edge types that the vertices of the link diagram represent. The parentheses around an edge type signify that its corresponding vertex lies in the interior of the link diagram either directly above or below the type. Edges with the same number of arrows identify to each other. Note from Figure 45 that after edge identifications are made each link diagram is a torus.

Now we may present Adams’ proof. First, note that since edges are identified in pairs the link diagram contains an even number $K$ of triangles. Next, let $z^1, z^2, \ldots, z^K$ be the variables associated with each triangle and $t_1, t_2, \ldots, t_k$ be the types of the edges (represented by vertices) in a given link diagram. Then, for each edge type (not all necessarily distinct, as in Figure 45) we have a function in terms of the $Z^1, z^2, \ldots, z^K$ that we set equal to 1. Now, suppose that we have satisfied $k - 1$ of these equations, leaving the equation for $t_k$ unsatisfied. Also, note that $z_1 z_2 z_3 = z \left( \frac{z-1}{z} \right) \left( \frac{1}{1-z} \right) = -1$ for every $z^i$. Then, multiplying
the left sides of the latter equations for each $z^i$ corresponding to a triangle in
the link diagram, we find that

$$(z_1^{i_1} z_2^{i_1} z_3^{i_1}) (z_1^{i_2} z_2^{i_2} z_3^{i_2}) \ldots (z_1^{K} z_2^{K} z_3^{K}) = 1$$ (1)

since $K$ is even. Then, since $k-1$ equations are satisfied, the product of functions
for the $t_{k-1}$ edge types must equal 1. Thus, (1) implies that the function for $t_k$
also equals 1. Therefore, for each cusp it suffices to check $k - 1$ edge equations,
dropping one equation from each cusp. Thus, considering all cusps we must
check $N - l$ edge equations. In our investigation there will always be $2(m + n - 3)$
edge types; we must therefore check $2(m + n - 3) - l$ edge equations.

Finally, we must show that the sum of the angles around a given edge in
the embedding of $\mathcal{P}$ (represented by vertices in the link diagram) is exactly $2\pi$.
Recall that after edge identifications are made the link diagram is torus. Then,note that since the edges of the triangles are identified in pairs a given torus will
have $K$ faces and $3K^2$ edges. Since the Euler characteristic of a torus is $\chi = 0$,
the number of vertices on the torus is $0 - K + \frac{3K^2}{2} = \frac{K}{2}$. Thus, the sum of angles
between edges on the torus is $\pi(2k)$. As the $k - 1$ edge equations are satisfied,
the sum of angles around the edge with type $t_k$ is $2k\pi - 2(k - 1)\pi = 2\pi$. Since
$t_k$ is arbitrary, the sum of angles around an edge of any type will equal $2\pi$.
Therefore, setting the product of edge variables about any edge of given type
equal to 1 will guarantee that the edge conditions are satisfied.

### 3.4 Satisfying the Completeness Conditions

Poincare’s theorem dictates that our embedding of $\mathcal{P}$ must satisfy completeness
conditions. Here, Adams cites Maskit in stating that we can confirm complete-
ness simply by checking that for each ideal vertex $x_0$ of $\mathcal{P}$ and any product of
face-identifying isometries that fix $x_0$, that this product is parabolic (i.e. fixes
only \( x_0 \). Adams then provides a lemma stating that we can simplify this process in the case where we have a 3-manifold with only tori in the boundary of its closure. Since \( S^3 - L \) is homeomorphic to the complement in \( S^3 \) of a tubular neighborhood of \( L \), \( S^3 - L \) is a 3-manifold with the aforementioned quality. Specifically, Adams’ lemma states that if the edge equations have been satisfied for \( \mathcal{P} \), we can confirm completeness by checking that one isometry from each cusp subgroup is parabolic. Once again, the proof is taken from Adams.

First, we will define what is meant by a cusp subgroup. Note that once we have \( z^1, z^2, \ldots, z^{2(m+n-3)} \) such that the edge equations are satisfied we can imbed \( \mathcal{P} \) into \( \mathbb{H}^3 \). Also, note that this embedding entirely specifies the face-identifying isometries. Thus, we have induced a homomorphism \( \phi \) of \( \pi_1(M) \) into \( \text{Isom}^+(\mathbb{H}^3) \), where \( M \) is the manifold realized by identifying all faces of \( \mathcal{P} \). Essentially, this means that the group formed by the set of equivalence classes of loops on the manifold is realized in \( \mathbb{H}^3 \) as a subset of the orientation-preserving isometries. Note once again that any other embedding of \( \mathcal{P} \) will be isomorphic to this one, implying that the two homomorphic images of \( \pi_1(M) \) in \( \text{Isom}^+(\mathbb{H}^3) \) will be conjugate via the given isometry.

Now, let \( T \) be a boundary torus in \( \hat{M} \). Then, \( \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z} \) and, since \( L \) is a nonsplittable link, \( \pi_1(T) \) injects into \( \pi_1(\hat{M}) \cong \pi_1(M) \). We then call \( \phi(\pi_1(T)) \) a cusp subgroup of \( \phi(\pi_1(M)) \). Next, if \( a_1 \) and \( b_1 \) generate \( \pi_1(T) \), \( \phi(a_1) \) and \( \phi(b_1) \) generate \( \phi(\pi_1(T)) \) and, since the fundamental group of a torus is Abelian, \( \phi(a_1)\phi(b_1) = \phi(b_1)\phi(a_1) \). Thus, the elements of \( \phi(\pi_1(T)) \) are all of the form \( \phi(a_1)^r\phi(b_1)^s \).

Next, suppose \( \phi(a_1)^r\phi(b_1)^s \) is parabolic with fixed point \( x_0 \). Then, since \( \phi(a_1)^r \) and \( \phi(b_1)^s \) commute they are either both loxodromic (fixing \( x_0 \) and another point) or both parabolic. If they are loxodromic, their product would be as well; thus, \( \phi(a_1)^r \) and \( \phi(b_1)^s \) are both parabolic. Note that this fact
applies even if either \( r = 0 \) or \( s = 0 \). Thus, the generators of \( \phi(\pi_1(T)) \) are parabolic and as such all of its elements are parabolic. Since we started with an arbitrary element of the cusp subgroup, checking that one element of \( \phi(\pi_1(T)) \) is parabolic suffices to show that every element is parabolic.

Now, note that if a face-identifying isometry preserving \( x_0 \) is parabolic it corresponds to a Euclidean translation in \( S^2_\infty \) when \( x_0 \) is at \( \infty \). Then, by Adams’ lemma we can ensure that a cusp is complete by showing that two identifying edges on the boundary of the cusp’s corresponding link diagram are Euclidean translates of one another. As with the edge equations, we accomplish this by constructing a function in terms of relevant edge variables and setting that function equal to a constant. In general, this process is more subjective because there is not necessarily one way of choosing relevant edge variables. Nevertheless, our specific triangulation of \( \mathcal{P} \) allows for a method that will always produce a cusp equation for each cusp subgroup.

Here, we want to specify two identifying edges on the boundary of a link diagram and a corresponding cusp equation such that one edge is a Euclidean translate of the other when the equation is satisfied. To accomplish this, we want two edges \( F_1 \) and \( F_2 \) (since edges in the link diagram represent faces of tetrahedra) such that there is a third edge \( F_3 \) that shares a distinct vertex with each of the first two edges and that \( F_1 \) and \( F_2 \) lie on the same side of the line containing \( F_3 \), as in Figure 46.

Let \( e_1 \) and \( e_2 \) be the vertices that \( F_3 \) shares with \( F_1 \) and \( F_2 \), respectively. Then, when we consider the triangles in the link diagram with vertices at \( e_1 \) and \( e_2 \), setting the product of corresponding edge variables equal to \(-1\) will ensure that \( F_1 \) and \( F_2 \) are Euclidean translates of one another. Specifically, setting the product of relevant edge variables equal to \(-1\) guarantees that the length of \( F_1 \) will equal that of \( F_2 \) and that the angles at \( e_1 \) and \( e_2 \) will have sum \( \pi M_2 \),
Figure 46: We can ensure that $F_1$ is the Euclidean translate of $F_2$ by setting the product of edge variables along their interior angles equal to $-1$.

$M_2 \in \mathbb{Z}$. As with the edge equations, we could show using the link diagram that $M_2 = 1$; i.e. the lines on which $F_1$ and $F_2$ lie will be parallel. A more detailed explanation for why this cusp equation guarantees that $F_1$ is a translate of $F_2$ is similar to (yet much more tedious than) that of the effectiveness of edge equations. Thus, such an explanation will not be presented here.

Now, we must choose two identifying faces on $\mathcal{P}$ for which an $F_3$ exists such that in the link diagram the edges corresponding to these faces relate in the manner just described. Note that generally there is no guarantee that these particular identifying edges lie on the boundary of the link diagram. We must therefore dictate there our choice of $F_1$ and $F_2$ lie on the boundary. For each link diagram, our choice of $F_1$ will be a face of either $\mathcal{T}_1$ or $\mathcal{T}_2$ projected to the bounding plane and $F_2$ will be the edge to which $F_1$ identifies. Depending on the choice of $x_0$ at $\infty$, these faces project to edges in the link diagram whose vertices represent a type 1 edge and a type 2 edge. Next, our triangulation of $\mathcal{P}$ guarantees that there will always exist a tetrahedron with three type 1 edges and another with three type 2 edges. Projections of these latter tetrahedra will produce edges whose vertices both represent type 1 and type 2 edges, respectively. Thus, an edge from either $\mathcal{T}_1$ or $\mathcal{T}_2$ will serve as $F_1$, its identifying partner.
face will serve as $F_2$, and one of the faces containing two edges of the same type will serve as $F_3$.

We will now show why the statements in the previous paragraph are true and specify the exact faces on $T_1$ and $T_2$ we wish to use. Recall that the tetrahedra $T_1$ and $T_2$ contain the vertices 1, $a$, $a+1$ and $a+b$ in their respective polyhedra. Next, recall that before digon collapse there were four edges on $\partial C''_1$ and $\partial C''_2$ that did not lie on a digon; these were the edges $(1,a+1)$, $(1,a+b)$, $(a,a+1)$ and $(a,a+b)$. Thus, in their respective polyhedra $T_1$ and $T_2$ contain these edges. Furthermore, in $\mathcal{P}_1$ the edges $(1,a+b)$ and $(a,a+1)$ have type 1 while $(1,a+1)$ and $(a,a+b)$ have type 2. In $\mathcal{P}_2$, these types are switched. Thus, about any vertex of $T_1$ and $T_2$ there is at least one edge of type 1 and at least one type 2 edge. Use Figures 39 and 47 (below) to verify these claims.

![Figure 47: The edges and types of $\mathcal{P}_1$ and $\mathcal{P}_2$ before triangulation.](image_url)

Next, we claim that in the link diagram all edges with vertices representing a type 1 edge and a type 2 edge come from faces of $T_1$ and $T_2$ and their identifying partners. This is useful because it will give us versatility in choosing $F_1$, making the process of deriving cusp equations easier. From above, it is clear if we choose any vertex of $T_1$ or $T_2$ at $\infty$ and project that tetrahedron to the bounding plane,
the resulting Euclidean triangle will contain at least one edge with endpoints representing edges of type 1 and 2. Then, in the link diagram the partner to that edge will also exist (either glued or not) and will represent the face to which the given face on $T_1$ ro $T_2$ identifies. Due to the way we triangulated $P$, no other tetrahedra will contain both a type 1 and a type 2 edge. As $m$ and $n$ get larger, the tetrahedra in $P_1$ and $P_2$ that do not share a face with $T_1$ and $T_2$ will contain mostly new edge types. Although I will not prove this claim outright, its truth is visually obvious when we consider $m, n > 3$; refer to Figure 41, where tetrahedra derived from the 43 and 34 rational links have the quality just described. Thus, these tetrahedra cannot project to edges with type 1 and 2 vertices in the link diagram. Therefore, every edge in the link diagram with vertices representing type 1 and 2 vertices comes from $T_1$, $T_2$, or a tetrahedron that glues to $T_1$ or $T_2$.

Finally, I will show that one of the tetrahedra created by triangulating $P_1$ contains three type 1 edges, while another contains three type 2 edges. I furthermore claim (but will not show) that triangulating $P_2$ has the same result. In the link diagram, we will derive our $F_3$’s from these tetrahedra. To begin, note that for the 22, 23 and 32 rational links, $T_1$ itself will constitute some of these tetrahedra (Figure 48).

Next, for $m, n \geq 3$ the tetrahedron in $P_1$ with vertices 1, 2, 3 and $a + b$ will have three type 1 edges (Figure 49. These edges are $e_1 = (1, a + b)$, $e_2 = (1, 2)$ and $e_3 = (2, 3)$. Note that $e_1$ and $e_2$ share vertex 1 and $e_2$ and $e_3$ share vertex 2. Also, recall that since these edges have the same type, they must be oriented either toward or away from the vertices they share. Note from Figure 49a that in this case $e_1$ and $e_2$ are oriented toward 1 while we orient $e_2$ and $e_3$ away from 2. Thus, if we put 1 at $\infty$ in the upper half-space model (49b) two type 1 edges of this tetrahedron will go out to $\infty$, while putting 2 at $\infty$ (49c) orients
two type 1 edges toward the bounding plane. Finally, we note similar qualities for the tetrahedron with vertices $a, a+b-2, a+b-1$ and $a+b$ in $\mathcal{P}_1$, which contains three type 2 edges, and for the counterparts of these tetrahedra in $\mathcal{P}_2$.

Finally, (finally!) I will show that given any edge $F_1$ on the boundary of a link diagram whose vertices represent a type 1 and a type 2 edge, there will be an $F_2$ also on the boundary that identifies with $F_1$ and an $F_3$ whose vertices are the equivalent vertices of $F_1$ and $F_2$. Then, taking a product of the edge variables corresponding to the interior angles of $F_1$ and $F_2$ (with respect to $F_3$) and setting that product equal to $-1$ will ensure that $F_2$ is a Euclidean translate of $F_1$. Then, at least one face-identifying isometry in the cusp subgroup corresponding to that link diagram will be parabolic. Since these properties will be true for all link diagrams, Adams' lemma implies that we will have satisfied the completeness conditions.

To accomplish this, we dictate that the link diagram be constructed in a more specific manner. First, let $x_0$ be a vertex on $\mathcal{T}_1$ and put $x_0$ at $\infty$ in the upper half-space model. Then, at least one of the vertical edges of $\mathcal{T}_1$ will have
type 1, while at least one other will have type 2. Let $e_1$ be one of these type 1 edges, $e_2$ be a type 2 edge and let $F_1$ be the vertical face containing $e_1$ and $e_2$. Now, glue the appropriate tetrahedron via the face-identifying isometry to the vertical face of $T_1$ containing $e_1$ that is not $F_1$. Then, continue to glue tetrahedra that contain $e_1$ to this structure via isometries that preserve $e_1$ such that $F_1$ remains unglued. Eventually, a tetrahedron $T_i$ containing three type 1 edges will be glued to this structure such that one of these edges is $e_1$ and one other is a vertical edge $e'_1$ with the same orientation as $e_1$. Figure 50 shows how this process progresses.

Now, let $F_3$ be the face on $T_i$ containing $e_1$ and $e'_1$. Moving forward, we would like to glue tetrahedra around $e'_1$ in the same manner as we had $e_1$ and in the same direction (clockwise/counterclockwise). In this way, we guarantee that we glue the tetrahedron $T'_1$ that identifies to $T_1$ via $F_1$ to the overall structure before $T_1$ is to be glued again. Now, let $F_2$ be the face of $T'_1$ that contains $e'_1$ and the edge $e'_2$ that identifies with $e_2$. Since $F_2$ can only be glued to $F_1$ in our structure, these faces remain unglued after $T'_1$ is glued to the structure. Finally,
Figure 50: We glue tetrahedra preserving $e_1$ via the face-identifying isometries.

continue to glue appropriate tetrahedra containing $\infty$ via isometries preserving $\infty$ until all faces on the boundary of this polyhedron identify in pairs. Projecting this structure to the bounding plane, we have a link diagram such that $F_1$ and $F_2$ lie on the boundary and $F_3$ contains their type 1 endpoints (Figure 51).

Figure 51: We glue tetrahedra around $e_1'$ in the same direction as we had $e_1$.

Thus, we have guaranteed that if we construct a link diagram in the specified manner we will always have the desired setup. Then, the equation for that cusp will consist of taking the product of edge variables at $e_1$ and $e_1'$ from triangles lying on the appropriate side of $F_3$ and setting that product equal to $-1$. Also, note that we just as well could have begun by gluing tetrahedra around $e_2$. Thus, for any $F_1$ that we choose and its corresponding $F_2$, there are two $F_3$'s (one containing type 1 edges and the other type 2 edges) that we could use.
to construct an equation for the cusp. This fact will be useful later when we
discuss which edge equations should be dropped from each cusp.

To conclude our discussion of edge equation we must tackle one more issue:
if \( L \) has two components, it will have two cusps and thus two distinct link
diagrams. To build each of these link diagrams from \( T_1 \) and/or \( T_2 \), we must
show that in this case all the vertices of \( T_1 \) and \( T_2 \) do not correspond to the
same cusp. Recall from Section 1 that \( L \) has two components if and only if \( m \)
and \( n \) are both odd. Now, consider \( \partial P_1 \) before triangulation. When \( m \) and \( n \)
are odd, the following properties hold at vertices 1, \( a \), \( a + 1 \) and \( a + b \):

1. The type of each edge is either 1 or 2.
2. Three edges meet at the vertex such that two of them have the same type.
3. The two edges with equal type are both oriented either toward or away
   from the vertex.
4. Type 1 and 2 edges have opposite orientation with respect to the vertex.
5. These are the only four vertices where type 1 and type 2 edges meet.

The same properties hold at the corresponding vertices of \( P_2 \). Refer to Figure
47 to verify these claims for \( L_0 \).

Now, let \( v_1 \) be a vertex toward which a type 1 edge is oriented. After
identifications are made, all such vertices will identify and thus correspond to
the same cusp. Now, let \( v_2 \) be a vertex toward which a type 2 edge is oriented.
Properties 4 and 5 above guarantee that even after identifications are made,
\( v_1 \) will never identify to \( v_2 \). Thus, these vertices must correspond to different
cusps. Finally, recall that 1, \( a \), \( a + 1 \) and \( a + b \) are the vertices of \( T_1 \) and \( T_2 \) in
their respective polyhedra. Thus, whenever \( m \) and \( n \) are odd there exist vertices
\( v_1 \) and \( v_2 \) on \( T_1 \) (or, equivalently, \( T_2 \)) such that \( v_1 \) and \( v_2 \) do not correspond to
the same cusp. Therefore, if \( L \) has two cusps we can build two link diagrams by initially sending \( v_1 \) to \( \infty \) in one and \( v_2 \) to \( \infty \) in the other.

3.5 The System of Equations

Thus far, our investigation has consisted of applying general concepts presented in Adams’ doctoral thesis to a specific subset of links. To this point, we have shown that given a rational link \( L \) of notation \( mn \), \( m, n \geq 2 \) we can develop a system of \( 2(m + n - 3) \) equations such that if the solutions to this system are complex numbers with positive imaginary part, \( S^3 - L \) admits a hyperbolic structure. However, we have exhausted much energy throughout this investigation in developing very specific ways to performing certain tasks (triangulating \( \partial P_1 \) and \( \partial P_2 \), assigning edge variables and building link diagrams). In this section, these efforts bear fruit: here, we show that we can determine the edge equations entirely from \( m \) and \( n \). Thus, in continuing to study the complements of links in \( L \) we can skip some of Adams’ constructions altogether.

First, let \( p(e_t) \) be the product of edge variables that are set to 1 in the equation for an edge \( e_t \) of type \( t \). Then, our procedures for triangulating \( \partial P_1 \) and \( \partial P_2 \) and for assigning edge variables guarantee that the following rules will hold for \( p(e_t) \) where \( t = 1 \) or \( t = 2 \).

1. \( p(e_1) \) contains \( (z^1_1)^2 \) and \( (z^2_1)^2 \)
2. \( p(e_2) \) contains \( (z^1_2)^2 \) and \( (z^2_2)^2 \)
3. if \( n = 2 \), \( p(e_1) \) contains \( (z^3_1) \)
4. if \( m = 2 \), \( p(e_2) \) contains \( (z^7_1) \)
5. if \( n > 2 \), \( p(e_1) \) contains \( (z^3_1)^2z^3_3 \) and \( (z^4_1)^2z^4_3 \)
6. if \( m > 2 \), \( p(e_2) \) contains \( (z^3_3)^{2(m+n-3)} - 1)^2z^2_1 \) and \( (z^3_3)^{2(m+n-3)} - 1)^2z^2_1 \)
7. if \( m > 2 \), \( p(e_1) \) contains \( z_1^{2n-1} \) and \( z_1^{2n} \)

8. if \( n > 2 \), \( p(e_2) \) contains \( z^{2n-3} \) and \( z^{2n-2} \)

9. if \( n > 3 \), \( p(e_1) \) contains \( (z_1^t)^2 \) for \( 3 \leq t \leq 2n - 2 \)

10. if \( n > 3 \), \( p(e_2) \) contains \( (z_3^t)^2 \) for \( 2n - 1 \leq t \leq 2(m + n - 3) \)

Note that each of these properties is essentially caused by how tetrahedra identify given a certain set of \( m \) and \( n \). Using these rules, we find the edge equations for the corresponding \( e_1 \) and \( e_2 \) of \( L_0 \):

\[
p(e_1) = (z_1^1)^2(z_3^3)^2(z_2^3)^2(z_1^4)^2z_1^3z_1^4z_1^6 = 1
\]

\[
p(e_2) = (z_2^1)^2(z_3^2)^2(z_3^3)^2z_1^6(z_3^2)^2z_1^3z_3^4 = 1
\]

Refer to Figure 45 to verify that these are correct. Further, note the following properties for \( p(e_t) \) when \( t \neq 1, 2 \).

1. if \( t \) odd, \( p(e_t) \) contains \( z_2^t \) and \( z_2^{t+1} \)

2. if \( t \) even \( p(e_t) \) contains \( z_2^{t-1} \) and \( z_2^t \)

3. if \( n > 3 \) and \( t < 2n - 2 \), \( p(e_t) \) contains \( z_3^{t+2} \)

4. if \( 2n - 2 < t < 2(m + n - 3) - 2 \), \( p(e_t) \) contains \( z_1^{t+2} \)

5. if \( t > 2n \) and \( t \) odd, \( p(e_t) \) contains \( z_1^{t-3} \)

6. if \( t > 2n \) and \( t \) even, \( p(e_t) \) contains \( z_1^{t-1} \)

7. if \( t = 2n - 3 \) or \( t = 2n - 1 \), \( p(e_t) \) contains \( z_1^3 \)

8. if \( t = 2n - 2 \) or \( t = 2n \), \( p(e_t) \) contains \( z_3^2 \).

Thus, the edge equations for the remaining edge types of \( L_0 \) are:
\[ p(e_3) = z_2^3 z_2^4 z_3^1 = 1 \]
\[ p(e_4) = z_2^3 z_2^4 z_3^2 = 1 \]
\[ p(e_5) = z_2^3 z_2^6 z_3^1 = 1 \]
\[ p(e_6) = z_2^5 z_2^6 z_3^2 = 1 \]

Finally, recall that we will our cusp equations will always involve a type 1 or 2 edge. When deciding which edge equation to drop from each cusp, it is always smart to get rid of the one with more variables. In any case, the cusp equation(s) used should always involve the type 1 or 2 edge whose equation was not dropped. Therefore, whenever \( L \) has two cusps both cusp equations will involve the same edge and an equation of an edge of type \( t > 2 \) will also have to be dropped. Furthermore, note that in all of these cases our system of \( 2(n + m - 3) \) equations reduces to \( m + n - 3 \) equations in the same number of variables. This occurs because \( z^{2k-1} = z^{2k} \) in all cases. Check this for \( L_0 \). Note that after solving for the system corresponding to \( L_0 \), \( z^i = .5 + .288675 \) for all \( i \). Thus, the tetrahedron corresponding to this link are all equivalent. Also, note that this value of \( z \) parametrizes Euclidean triangles with angles \( \frac{\pi}{6}, \frac{\pi}{6} \) and \( \frac{2\pi}{3} \).

Up to this point, the results of Adams’ doctoral thesis ensure that if our system of equations yields a set of complex numbers all with positive imaginary part, \( S^3 - L \) is hyperbolic. However, in general we do not knot that this system will always yield such a solution, and we have not determined that this will be the case for all \( L \in \mathcal{L} \). Nonetheless, Adams provides a lemma stating that if we know that a given link is hyperbolic, our system of equations yields a possibly singular solution from which a correct hyperbolic volume can always be derived. I will not present the proof of this lemma. It then remains to show that all \( L \in \mathcal{L} \) are hyperbolic. In [2], Adams presents the following fact proven by Menasco: “Prime non-splittable alternating links that are not 2-braids are hyperbolic”. Fortunately, every link in \( \mathcal{L} \) falls in this category.
To see why, first recall that from Section 1 that all \( L \) is alternating. Second, we argue that \( L \) is nonsplittable. Imagine that we twist two horizontal strands into a tangle \( T_1 \) of notation \( n \), then glue its north ends together. Considering previous constructions, we can easily see that this produces either a knot or a nonsplittable link. Now imagine that we first twisted the strands into a tangle \( T_2 \) of notation \( m \), then reflected its projection as in Section 1. If after twisting the right ends of \( T_2 \) into a tangle of notation \( n \) the resulting link has two components, we still will not be able to split those components. Perhaps a more precise way to say this is that we do not have sufficient freedom of twisting in the \( m \) tangle to undo the linking that occurs in the \( n \) tangle.

Next, we must show that \( L \) is prime. In [2], Adams credits Menasco with proving that “an alternating link is composite if and only if every reduced alternating projection has a circle that crosses the link twice and has crossings to either side [(inside/outside)]”. First, note that as with the natural numbers, a link is composite if it is not prime. Also, an alternating projection is reduced if it does not contain any crossings like the one in 52. Now, if we draw a circle around any crossing or set of crossings in the projection of \( L \) such that crossings lie both inside and outside of the circle, this circle will always intersect the link in at least four places (Figure 53). Therefore, \( L \) is not composite and, as such, must be prime.

![Figure 52: A link projection is reduced if it does not contain a crossing like the one shown here.](image)

Finally, in Section 1 we constructed the rational links in \( L \) such that none of these links are 2-braids. Here, we can construct any 2-braid (Figure 54, from
Figure 53: Every circle that contains crossings to either side intersects the link in at least 4 places.

Using the method in Section 1 by dictating that $m^*$ be nonzero and that $n^*$ equal 1 or $-1$ such that the sign of $m^*$ and $n^*$ are equivalent. However, Adams notes that in this case an edge of given type would collapse to itself during digon collapse. Furthermore, after all digons have been collapsed the resulting structure would be either an annulus or a Mobius band. As we have already shown that after collapsing all digons $C''_1$ becomes a polyhedron with four faces, $L$ must not be a 2- braid. Thus, from Menasco’s results we can conclude that all $L \in \mathcal{L}$ are hyperbolic.

3.6 Determining the Hyperbolic Volume

Upon solving the system of equations, we have determined (up to isometry) the hyperbolic structure of $L$. Specifically, to each tetrahedron $T_i$ in $\mathcal{P}$ there corresponds a complex constant that we can use to determine the dihedral angles of $T_i$ and thus $T_i$ itself. Also, these constants specifically determine how we must
embed $\mathcal{P}$ such that isometric images of this embedding cover $\mathbb{H}^3$. In [2], Adams cites the Mostow Rigidity theorem in stating that if the volume associated with a 3-manifold is finite, this hyperbolic structure is unique. In this investigation, we have modeled the complements of hyperbolic links using ideal tetrahedra, which have finite volume [4].

Thus, we could determine whether two links in $\mathcal{L}$ are homeomorphic by examining their hyperbolic structures. However, for large $m$ and $n$ this would clearly be tedious; we would like a more concise way to perform this comparison. Fortunately, we can find the volume of a given tetrahedron using only its dihedral angles. We then determine the hyperbolic volume of $S^3 - L$ by summing the volumes of the component tetrahedra of $\mathcal{P}$. Since the hyperbolic structure of $S^3 - L$ is unique up to isometry, any isometric manifestation of this structure will produce the same dihedral angles and thus the same volume. Hyperbolic volume is thus said to be invariant. Thus, an immediate implication is that if we derive different volumes from the complements of two links, those links cannot be homeomorphic (topologically equivalent). We will now present a method for finding the hyperbolic volume of $L$ and discuss trends that emerge when we consider links of increasingly high $m$ and $n$.

First, we must determine the dihedral angles of each $\mathcal{T}_i$ in $\mathcal{P}$ using the
solutions to our system of equations. Let \( z^i \) be the complex constant that parametrizes the angles of \( T_i \). Then, we consider a Euclidean triangle in the complex plane with vertices at 0, 1 and \( z^i \) and use trigonometry to determine the angles. Specifically, we draw a line segment from \( z^i \) to the real axis such that this line is perpendicular to the real axis. Then, we can determine the length of this segment and that of the triangle’s edges and use inverse trigonometric functions to determine the angles. As this process is somewhat elementary, we will not elaborate on it here.

Next, Adams provides a method for determining the volume of an ideal tetrahedron. Let \( T_i \) be such a tetradron with dihedral angles \( \alpha, \beta \) and \( \gamma \). First, note that in the upper-half plane model of hyperbolic 2-space an arc \( \Gamma = z(t) = x(t) + iy(t), c \leq t \leq d \) has length \( l(\Gamma) = \int_a^b \frac{|z(t)|}{y(t)} \, dy \). One can thus see that where vertical geodesics seem to be parallel in this model, they actually become closer together as they approach \( \infty \) (i.e. as \( y \) increases). With this in mind, it is no stretch to state that in \( \mathbb{H}^3 \) we must integrate \( \frac{1}{z} \) over the entire region of \( T_i \) in order to find its volume. Here want to send one vertex of \( T_i \) to \( \infty \) such that the tetrahedron has three vertical faces and fourth face lying on the hemisphere given by \( z = \sqrt{1 - x^2 - y^2} \). Then, projecting \( T_i \) to the bounding plane produces a Euclidean triangle whose vertices lie on the unit circle. If we then draw in the unit circle we note that the triangle and circle share a center point. Thus, we cut the triangle into 6 right triangles as in Figure 55 (provided by Adams in [4]). Note that the labeling of the angles is a fact from geometry that can be derived when we consider the isosceles triangles that compose the larger triangle.

Now, in the shaded triangle we want to align the edge adjacent to \( \alpha \) with the real axis. We will now find the volume of the section of \( T_i \) corresponding to this triangle. Note in the shaded region \( x \) is bounded by 0 and \( \cos \alpha \) and \( y \) is
Figure 55: We can cut any triangle inscribed in a circle such that the angles with the same labels all have equal measure.

bounded by 0 and \( x \tan \alpha \). Furthermore, \( z \) is bounded below by \( \sqrt{1 - x^2 - y^2} \) and above by \( \infty \). Now, we must integrate \( \frac{1}{z^3} \) over these bounds with respect to \( z, y \) and \( x \) (in that order). Thus, the volume \( V(\alpha) \) of this region is given by

\[
V(\alpha) = \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \int_0^{\infty} \frac{1}{\sqrt{1 - x^2 - y^2}} \frac{1}{z^3} \, dz \, dy \, dx
\]  

(2)

With some calculus handiwork (provided by Adams), we simplify the right sight
of (2) down to

\[
V(\alpha) = \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \int_{\sqrt{1-x^2-y^2}}^{\infty} \frac{1}{z^3} \, dz \, dy \, dx
\]

= \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \frac{1}{2(1-x^2-y^2)} \, dy \, dx

= \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \frac{1}{4\sqrt{1-x^2}} \left( \frac{1}{\sqrt{1-x^2}-y} + \frac{1}{\sqrt{1-x^2}+y} \right) \, dy \, dx

= \int_0^{\cos \alpha} \frac{1}{4\sqrt{1-x^2}} \ln \left( \frac{\sqrt{1-x^2+x \tan \alpha}}{\sqrt{1-x^2-x \tan \alpha}} \right) \, dx

= \int_0^{\cos \alpha} \frac{1}{4\sqrt{1-x^2}} \ln \left( \frac{\sqrt{1-x^2 \cos \alpha + x \sin \alpha}}{\sqrt{1-x^2 \cos \alpha - x \sin \alpha}} \right) \, dx.

Now, let \( x = \cos \theta \). Then \( \sqrt{1-x^2} = \sin \theta \) and \( dx = -\sin \theta \, d\theta \). Making these substitutions, we obtain

\[
\frac{1}{4} \int_{\alpha}^{\pi/2} \ln \left( \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)} \right) \, d\theta
\]

Now, letting \( u = \theta - \alpha \), we obtain \( du = d\theta \) and

\[
V(\alpha) = \frac{1}{4} \int_0^{\pi/2 - \alpha} \ln \left( \frac{\sin(u + 2\alpha)}{\sin(u)} \right) \, du
\]

Adams dictates that we use this function as is. Then, the volume of \( T_i \) is given by

\[
\text{Vol}(T_i) = 2V(\alpha) + 2V(\beta) + 2V(\gamma).
\]

Finally, the hyperbolic volume of \( H \) of the complement of a rational link \( L \) with notation \( mn \) is

\[
H(S^3 - L) = \sum_{i=1}^{2(m+n-3)} \text{Vol}(T_i)
\]

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where $T_i$ is a tetrahedron in the model $P$ of the link complement.

Recall that the dihedral angles for all tetrahedra corresponding to the complement of $L_0$ are $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{6}$, and $\gamma = \frac{2\pi}{3}$. Then, applying Adams’ equation to $S^3 - L_0$ we find that $\text{Vol}(T_i) = .6766$. Since the tetrahedra corresponding to $L_0$ are equivalent, $H(S^3 - L_0) = 6(.6766) = 4.0598$. Table 3.6 displays the hyperbolic volumes of the complements of rational links with notation $mn$, $2 \leq m, n \leq 5$.

Table 3.6: Hyperbolic Volumes for Small $m, n.$

<table>
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<tr>
<th></th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 2$</td>
<td>2.0298</td>
<td>2.8281</td>
<td>3.1640</td>
<td>3.3317</td>
</tr>
<tr>
<td>$m = 3$</td>
<td>2.8281</td>
<td>4.0598</td>
<td>4.5921</td>
<td>4.8512</td>
</tr>
<tr>
<td>$m = 4$</td>
<td>3.1640</td>
<td>4.5921</td>
<td>5.2387</td>
<td>5.5565</td>
</tr>
<tr>
<td>$m = 5$</td>
<td>3.3317</td>
<td>4.8512</td>
<td>5.5565</td>
<td>5.9079</td>
</tr>
</tbody>
</table>

First, note that Table 3.6 reflects what the trends in the system of equations show: there is plenty of symmetry within this subset of links. Here, links with notation $mn$ and $nm$ have the same hyperbolic volume. In the future, examining these links using other invariants may be useful. Also, when we consider links of distinct notation such that $m$ and $n$ are not interchanged, we notice that these links have different volume. Note that this occurs even when considering links with the same crossing number $m + n$. Thus, it seems that our subset of links truly is infinite, but perhaps not as infinite as our notation would suggest.

Finally, in his doctoral thesis Adams discusses an upper bound for the hyperbolic volume: specifically, if $M$ is the number of edge types in a triangulation, the hyperbolic volume will not exceed $1.014M$ ($1.014$ is the volume of a single regular ideal tetrahedron; we see that the complement of the figure-eight knot contains two of these). However, Table 3.6 indicates that this upper bound could be improved. Note that moving either down or right across this table, the rate of
increase for the volume becomes smaller and smaller. Perhaps, then, there is an upper bound on the volumes corresponding to the entire subset. This ends our investigation.

References


