Quantum Time: Time as a Dynamical Variable

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Abstract

We will outline a formalism that treats time and space on equal footing as in special relativity. We will define an extended classical theory that treats time as a dynamical variable like the spatial coordinates. We will use canonical quantisation to replace Poisson brackets with commutators and show how a generator of time translations can be defined using the commutation relationship in quantum mechanics. Using this new formalism, we will derive Schrödinger's equation, Ehrenfest theorem, the propagator and expectation values of time for a simple harmonic oscillator potential.
"Time and quantum mechanics have, each of them separately, captivated scientists and laymen alike, as shown by the abundance of popular publications on "time" or on the many quantum mysteries or paradoxes. We too have been seduced by these two topics, and in particular their combination. Indeed, the treatment of time in quantum mechanics is one of the important and challenging open questions in the foundations of quantum theory." (Muga, Sala Mayato, and Egusquiza, 2002.)
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1 Introduction

The roles of time and space have always been different in quantum mechanics. Time is treated as a external parameter and the spacial coordinates are treated as dynamical variables. However in special relativity time and space are treated on an equal footing. Therefore there is a conflict between these two theories.

We will start with classical mechanics and define an extended theory with time as a variable. Each of the coordinates including time is a function of some monotonically varying parameter defined as $\beta$. We will outline the new Lagrangian mechanics and Hamiltonian mechanics. We will promote Poisson brackets to commutators using canonical quantization and use it to find a time translational operator. This allows us to find momentum in the spacetime representation. This new representation allows us to get the Schrödinger equation, Ehrenfest theorem and the propagator. We will also give a brief history of how time was treated in quantum mechanics by some of the founders of modern quantum theory.
2 Lagrangian Mechanics

Starting with traditional Lagrangian and Hamiltonian mechanics, a formalism will be created that treats time as a dynamical variable like the spatial coordinates. All coordinates depend on a monotonic variable, $\beta$. This new way of handling the Lagrangian and the Hamiltonian is called extended theory. In this chapter, we will briefly discuss the basics of Lagrangian mechanics. We will reintroduce generalized coordinates and review the Lagrange's equation.

A mechanical system is described by a set of $D$ Cartesian coordinates $x_i = x_1, x_2, \ldots, x_D$. The set $q_i = q_1, q_2, \ldots, q_D$ of $D$ independent variables is called the set of generalized coordinates. This set of variables fully specify the system. The transformation from the set of cartesian coordinates to generalized coordinates can be written as

$$q_i = q_i(x_1, x_2, \ldots, x_D, t).$$ (2.1)

The inverse transformation can be written as

$$x_i = x_i(q_1, q_2, \ldots, q_D, t).$$ (2.2)

The necessary and sufficient condition for the existence of the inverse
transformation is that the determinant of the Jacobian is non-zero.

\[ \det \left| \frac{\partial x(q,t)}{\partial q} \right| \neq 0 \] (2.3)

Lagrange's equation in generalized coordinates can be written as

\[ \frac{d}{dt} \left( \frac{\partial L(q,q',t)}{\partial q'} \right) - \frac{\partial L(q,q',t)}{\partial q} = 0 \] (2.4)

where \( L = T - V \) and we assume that the system only consists of potential forces. The momentum is given by

\[ p_i = p_i(q,q',t) = \frac{\partial L(q,q',t)}{\partial q'_i}. \] (2.5)

3 Hamiltonian Mechanics

This section will review some details of Hamiltonian Mechanics.

3.1 Hamilton Equations

The Hamiltonian is obtained by applying a Legendre transformation to the Lagrangian. The transformation from a function \( f = f(x,y) \) to a function \( g = g(x,w) \) is called a Legendre transformation. Let the partial derivatives of \( f \) be defined as

\[ \frac{\partial f(x,y)}{\partial x_i} = u_i(x,y); \quad \frac{\partial f(x,y)}{\partial y_j} = w_j(x,y) \]
where \( i = 1, \ldots, M \) and \( j = 1, \ldots, N \). Also, the partial derivatives of \( g \) are defined by

\[
\frac{\partial g(x, w)}{\partial x_i} = -u_i(x, w); \quad \frac{\partial g(x, w)}{\partial w_j} = y_j(x, w)
\]

for \( i = 1, \ldots, M \) and \( j = 1, \ldots, N \). The roles of \( w \) and \( y \) are interchanged for the last partial derivatives except for the minus sign. We get the Legendre transformation by defining \( g \) to be

\[
g(x, y) = \sum_i y_i \frac{\partial f}{\partial y_i} - f(x, y). \tag{3.1}
\]

\( g \) can also be expressed in its correct set of variable \( w \) but we can already use this transformation on the Lagrangian. When applied on the Lagrangian we obtain the Hamiltonian.

\[
H(q_i, p_i, t) = \sum_i \dot{q}_i \frac{\partial L(q_i, q'_i, t)}{\partial q'_i} - L(q_i, q'_i, t) = \sum_i q'_i p_i - L(q_i, q'_i, t) \tag{3.2}
\]

The Hamiltonian is the difference between \( 2T = 2p_i^2/2m = p_i m q'_i/m = q'_i p_i \) and the Lagrangian \( L = T - V \). We can easily see that the Hamiltonian represents the total energy of the system \( H = 2T - (T - V) = T + V \). The total differential of the Hamiltonian is

\[
dH = \sum_i \left( \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt. \tag{3.3}
\]
Expanding this further, we get

\[ dH = \sum_i \left( \frac{\partial (q_i p_i)}{\partial q_i} dq_i' + \frac{\partial (q_i p_i)}{\partial p_i} dp_i - \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial q_i} dq_i' \right) - \frac{\partial L}{\partial t} dt \]

\[ = \sum_i (p_i dq_i' + q_i' dp_i - p_i' dq_i + p_i dq_i') - \frac{\partial L}{\partial t} dt \]

\[ = \sum_i (q_i' dp_i - p_i' dq_i) - \frac{\partial L}{\partial t} dt. \tag{3.4} \]

where we know from Lagrange's equations that \( p_i = \frac{\partial L}{\partial q_i} \) and \( p_i' = \frac{\partial L}{\partial q_i}. \)

Comparing this with 3.3, the Hamilton equations are obtained. These are

\[ q_i' = \frac{\partial H}{\partial p_i} \quad p_i' = -\frac{\partial H}{\partial q_i} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3.5} \]

### 3.2 Poisson Brackets

The total derivative of a function \( f = f(q, p, t) \) is

\[ \frac{df}{dt} = \sum_i \left( \frac{\partial f(q, p, t)}{\partial q_i} q_i' + \frac{\partial f(q, p, t)}{\partial p_i} p_i' \right) + \frac{\partial f(q, p, t)}{\partial t} \]

\[ = \sum_i \left( \frac{\partial f(q, p, t)}{\partial q_i} \frac{\partial H(q, p, t)}{\partial p_i} - \frac{\partial f(q, p, t)}{\partial p_i} \frac{\partial H(q, p, t)}{\partial q_i} \right) + \frac{\partial f(q, p, t)}{\partial t} \]

\[ = \{f, H\} + \frac{\partial f(q, p, t)}{\partial t}. \tag{3.6} \]
where we used Hamilton's equations. We define the Poisson brackets for two functions $f = f(q, p, t)$ and $g = g(q, p, t)$ as

$$\{f, g\} = \sum_i \left( \frac{\partial f(q, p, t)}{\partial q_i} \frac{\partial g(q, p, t)}{\partial p_i} - \frac{\partial f(q, p, t)}{\partial p_i} \frac{\partial g(q, p, t)}{\partial q_i} \right) \quad (3.7)$$

Some useful identities when working with Poisson brackets are

$$\{f, f\} = 0 \quad (3.8)$$

$$\{f, g\} = -\{g, f\} \quad (3.9)$$

$$\{f, (\alpha g + \beta h)\} = \alpha \{f, g\} + \beta \{f, h\} \quad (3.10)$$

$$\{f, g h\} = g \{f, h\} + \{f, g\} h \quad (3.11)$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \quad (3.12)$$

When applying the Poisson brackets to the Hamiltonian, the following identities are obtained.

$$q'_i = \{q_i, H\} \quad p'_i = \{p_i, H\} \quad H' = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad (3.13)$$

$$\{q_i, q_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij} \quad \{p_i, p_j\} = 0 \quad (3.14)$$
4 Lagrangian Mechanics with Time as a Variable

We will now introduce a theory where the Lagrangian and Hamiltonian are defined in their extended forms. In this theory, we treat time as another variable like the spatial variables. The zeroth generalized coordinate is defined as time multiplied by the speed of light, \( c, q_0 = ct \). Its conjugate momentum \( p_0 \) is defined as the negative of the traditional generalized energy function \( H \) divided by \( c \). \( q_0 = ct \) was chosen so that all \( q_\mu \) have the unit length. \(-H/c\) was chosen so that all \( p_\mu \) have the unit \( kg/m/s \), the unit of momenta. For the sake of simplicity, we will say that \( c = 1 \). Therefore, \( q_0 = t \) and \( p_0 = -H \).

In this theory, there are \((D + 1)\) extended Lagrange’s equations. The extra Lagrange’s equation is equivalent to the traditional generalized energy theorem. This notation allows to use a four vector for every particle.

\[
q_0 = t \quad \quad p_0 = -H \quad \quad (4.1)
\]

In the General Parametric Method, a path is specified parametrically. Therefore each of the coordinates will be a function of some monotonically varying parameter \( \beta \). We will denote a derivative with respect to \( \beta \) as a dot over a quantity.
The parameter $\beta$ parametrizes all coordinates, including the time, and all momenta, including $p_0$. The parameter $\beta$ is initially unspecified. It will only be specified after all partial derivatives have been taken and the final differential equations of motion have been obtained. $\beta$ can then be chosen as any random, monotonic function. It is usually chosen such that it makes the differential equation of motion as simple as possible. In special relativity the monotonically varying parameter can be defined as the proper time.

We can now define the four vector notation.

\begin{equation}
q_\mu = q_0, q_1, q_2, \ldots, q_D \quad q_i = q_1, q_2, \ldots, q_D \quad (4.4)
\end{equation}

\[ p_\mu \text{ and } p_i \text{ are defined in the same way.} \]
4.1 Extended Lagrangian

The traditional Lagrangian can now be written as

\[ L = L(q_\mu, q_\mu') \]  \hspace{1cm} (4.5)

where \( q_\mu \) includes the time as well as the other coordinates.

The extended Lagrangian can be obtained by using Hamilton's principle. Hamilton's principle acts in the space of Lagrangian variables \( q, p \) and \( t \).

The form of the equation to find extremum paths of line integrals looks very similar to the Lagrange's equation. The line integral to find the extremum path is defined as

\[ I = \int_{t_1}^{t_2} f(q, q', t) \, dt. \]  \hspace{1cm} (4.6)

This is called the Action Integral. Hamilton's principle states that the natural path of motion makes the action integral an extremum.

We can now use Hamilton's Principle to obtain the extended Lagrangian.

\[ I = \int_{t_1}^{t_2} L(q_\mu, q'_\mu) \, dt = \int_{\beta_1}^{\beta_2} L \left( q_\mu, \frac{q'_\mu}{t} \right) \, \beta \, d\beta = \int_{\beta_1}^{\beta_2} L(q_\mu, q_\mu) \, d\beta. \]  \hspace{1cm} (4.7)

where \( q'_\mu \) is written as \( \frac{dq_\mu}{dt} \) and the limits of integration are changed by writing \( dt = \beta d\beta \).
Therefore, the extended Lagrangian can be defined as

\[ \mathcal{L}(q_\mu, \dot{q}_\mu) = \mathcal{L}(q_\mu, \frac{\dot{q}_i}{t}) = q_0 \mathcal{L}(q_\mu, \frac{\dot{q}_i}{q_0}). \] (4.8)

### 4.2 Extended Momenta

The generalized momenta in the extended theory are defined as

\[ p_\mu = p_\mu(q_\nu, \dot{q}_\nu) = \frac{\partial \mathcal{L}(q_\nu, \dot{q}_\nu)}{\partial \dot{q}_\mu}. \] (4.9)

For the extra zero-th component of the generalized momenta,

\[
\begin{align*}
p_0(q_\mu, \dot{q}_\mu) &= \frac{\partial \mathcal{L}(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} = \frac{\partial \mathcal{L}(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_i} = \mathcal{L}(q_\mu, \dot{q}_i) + q_0 \sum_{j=1}^{D} \frac{\partial \mathcal{L}(q_\mu, q_i')}{\partial q_j} \frac{\delta q_i}{\delta q_0} \\
&= \mathcal{L}(q_\mu, q_i') - q_0 \sum_{j=1}^{D} p_j(q_\mu, q_i') \frac{\dot{q}_j}{q_0} = \mathcal{L}(q_\mu, q_i') - \sum_{j=1}^{D} p_j(q_\mu, q_i') q_j \\
&= -H(q_\mu, q_i'). \quad (4.10)
\end{align*}
\]

Hence,

\[ \frac{\partial \mathcal{L}(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} = p_0(q_\mu, \dot{q}_\mu) = -H(q_\mu, q_i'). \] (4.11)

Therefore the zero-th component of the generalized momenta in the extended theory is the negative of the traditional generalized energy function \( H \).

The extended Lagrangian is homogeneous of degree one in the set of
generalized velocities \( \dot{q} \). We see this by considering

\[
L(q, \lambda \dot{q}) = \lambda \dot{q}_0 L(q_0, \frac{\lambda \dot{q}_0}{\lambda \dot{q}_0}) = \lambda L(q, \dot{q}).
\]

Using the definition of homogeneity in [8], \( L(q, \dot{q}) \) is homogeneous of degree one. This implies that

\[
L(q_\mu, \dot{q}_\mu) = \sum_{\nu=0}^{D} \dot{q}_\nu \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_\mu}.
\]

Using the definition in (4.9), we can then write the extended Lagrangian in terms of generalized velocities and momenta.

\[
L(q_\mu, \dot{q}_\mu) = \sum_{\nu=0}^{D} p_\nu(q_\mu, \dot{q}_\mu) \dot{q}_\nu. \tag{4.12}
\]

### 4.3 Extended Lagrange equations

The extended Lagrange's equations combine the traditional Lagrange's equations and the generalized energy theorem. The extended Lagrange's equation is

\[
\frac{d}{d\beta} \left( \frac{\partial L(q_\nu, \dot{q}_\nu)}{\partial \dot{q}_\mu} \right) - \frac{\partial L(q_\nu, \dot{q}_\nu)}{\partial q_\mu} = 0. \tag{4.13}
\]

The extended Lagrange's equations for \( \mu \neq 0 \) are analogous to the traditional Lagrange's equations. The extended Lagrange's equation for \( \mu = 0 \) is equivalent to the generalized energy theorem. Let us first look at
the case of $\mu \neq 0$.

\[
\frac{d}{dt} \left( \frac{\partial L(q,\dot{q})}{\partial \dot{q}_j} \right) - \frac{\partial L(q,\dot{q})}{\partial q_j} = 0
\]  

(4.14)

If we multiply this by $\dot{q} = \frac{dT}{d\beta}$, we get

\[
\frac{d}{d\beta} \left( \frac{\partial L(q,\dot{q})}{\partial \dot{q}_j} \right) - \dot{q} \frac{\partial L(q,\dot{q})}{\partial q_j} = 0.
\]  

(4.15)

This is equivalent to (4.13) as

\[
-\dot{q} \frac{\partial L(q,\dot{q})}{\partial t} = -c \frac{\partial L(q,\dot{q})}{\partial q_0}
\]  

(4.16)

and

\[
\frac{\partial L(q,\dot{q})}{\partial \dot{q}_i} = \frac{\partial L(q,\dot{q})}{\partial q_i'}
\]  

(4.17)

from (4.11). Now let's look at the case of $\mu = 0$. We begin with the generalized energy theorem.

\[
\frac{\partial H(q,\dot{q})}{\partial t} = -\frac{\partial L(q,\dot{q})}{\partial t}
\]  

(4.18)

Multiplying by $\dot{q} = \frac{dT}{d\beta}$ gives

\[
\frac{\partial H(q,\dot{q})}{d\beta} = -\dot{q} \frac{\partial L(q,\dot{q})}{\partial t}
\]  

(4.19)
Using eq. (4.11) we find

\[ \frac{\partial H(q_\nu, q_\lambda)}{d\beta} = \frac{d}{d\beta} \left( -\frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} \right) \]  

(4.20)

The right hand side of (4.19) gives us

\[ -t \frac{\partial L(q_\nu, q_\lambda)}{\partial t} = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial q_0} \]  

(4.21)

Therefore

\[ \frac{d}{d\beta} \left( -\frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial \dot{q}_0} \right) = \frac{\partial L(q_\mu, \dot{q}_\mu)}{\partial q_0} \]  

(4.22)

which is equivalent to (4.13).

5 **Hamiltonian Mechanics with Time as a Coordinate**

The traditional Hamilton’s equations will be combined into one set of extended Hamilton’s equations where time is treated as a coordinate.

The extended set of phase-space variables includes the new coordinate \( q_0 = t \) and the new momentum \( p_0 = -H \). The phase-space is then \( (2D + 2) \) dimensional, with the canonical coordinates

\[ q, p = q_0, q_1, \ldots, q_D, p_0, p_1, \ldots, p_D. \]  

(5.1)
The equations of motion in the extended Hamiltonian theory specify the phase-space trajectory as functions of the new parameter $\beta$, $q_{\mu} = q_{\mu}(\beta)$ and $p_{\mu} = p_{\mu}(\beta)$, include the two new coordinates $q_0$ and $p_0$.

5.1 From Traditional to Extended Hamiltonian Mechanics

In the extended Hamiltonian theory, all variables of phase-space are treated equally. Therefore we now have a new pair of canonically conjugate coordinates $(q_0, p_0)$.

In the extended Lagrangian theory, the momenta, $p_\mu$, are derived quantities. In the extended Hamiltonian theory, they are independent coordinates in phase-space. Thus we can no longer use the Lagrangian identities, (4.9), in the Hamiltonian theory.

We want Hamilton's equation to have the same form as in classical theory. Let the extended Hamiltonian be $\mathcal{K}$. Then we want the form of Hamilton's equations to be

$$\dot{q}_\mu = \frac{\partial \mathcal{K}(q_\nu, p_\nu)}{\partial p_\mu} \quad \dot{p}_\mu = -\frac{\partial \mathcal{K}(q_\nu, p_\nu)}{\partial q_\mu}.$$  \hspace{1cm} (5.2)

$\mathcal{K}$ is then chosen to be

$$\mathcal{K}(q_\mu, p_\mu) = p_0 + H(q_\mu, p_\mu) \hspace{1cm} (5.3)$$

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We see from eq. (4.1) that

$$K(q_\mu, p_\mu) = 0.$$  

(5.4)

$K$ could also be chosen as a non-zero constant because only the derivatives of $K$ are going to be of interest. For the sake of simplicity we will say that $K$ is equal to zero. This condition, $K(q_\mu, p_\mu) = 0$, must not be applied until all partial derivatives have been applied.

5.2 Equivalence to Traditional Hamilton Equations

The extended Hamiltonian $K(q_\mu, p_\mu)$ is defined as in eq. (5.3), and the extended Hamilton equations are defined in eq. (5.2). When $\mu = 0$, the extended Hamilton equations are equivalent to the traditional form of the generalized energy theorem. And when $\mu \neq 0$, they are equivalent to the traditional Hamilton's equations.

Both, the time $t$ and the parameter $\beta$, vary monotonically along the path in the configuration space. Therefore, the derivative $\frac{dt}{d\beta}$ can never be zero.

Letting $\mu = 0$ in (5.2), we find $q_0 = \frac{d\beta}{d\beta} = 1$. Using this and (5.3), we can
prove equivalence for the $\mu \neq 0$ case.

$$\dot{q}_i = \frac{dt}{d\beta} q'_i = q_i = \frac{\partial K(q_\mu, p_\mu)}{\partial p_i} = \frac{\partial H(q_\mu, p_\mu)}{\partial p_i}$$  \hspace{1cm} (5.5)

$$\dot{p}_i = \frac{dt}{d\beta} p'_i = p_i = -\frac{\partial K(q_\mu, p_\mu)}{\partial q_i} = -\frac{\partial H(q_\mu, p_\mu)}{\partial q_i}$$  \hspace{1cm} (5.6)

For the case of $\mu = 0$,

$$\dot{q}_0 = \frac{dt}{d\beta} q'_0 = q'_0 = \frac{\partial K(q_\mu, p_\mu)}{\partial p_0} = 1$$  \hspace{1cm} (5.7)

$$\dot{p}_0 = \frac{dt}{d\beta} p'_0 = p'_0 = -\frac{\partial K(q_\mu, p_\mu)}{\partial q_0} = -\frac{\partial H(q_\mu, p_\mu)}{\partial q_0} = \frac{\partial H(q_\mu, p_\mu)}{\partial t}.$$  \hspace{1cm} (5.8)

5.3 Poisson Brackets with Time as a Coordinate

In extended Hamiltonian mechanics, all quantities of physical interest are assumed to be expressed as functions of the phase-space variables. For example,

$$f = f(q_\mu, p_\mu) = f(q_0, q_1, \ldots, q_D, p_0, p_1, \ldots, p_D)$$  \hspace{1cm} (5.9)

In the extended theory, the extended Poisson bracket of two phase-space functions $f(q_\mu, p_\mu)$ and $g(q_\mu, p_\mu)$ can be defined in the same way as (3.7) but with additional variables, $q_0$ and $p_0$.

$$\{f, g\} = \sum_{\mu=0}^{P} \left( \frac{\partial f(q_\nu, p_\nu)}{\partial q_\mu} \frac{\partial g(q_\nu, p_\nu)}{\partial p_\mu} - \frac{\partial f(q_\nu, p_\nu)}{\partial p_\mu} \frac{\partial g(q_\nu, p_\nu)}{\partial q_\mu} \right).$$  \hspace{1cm} (5.10)

The same identities as defined in eq. (3.8) to eq. (3.12) hold for the Poisson
brackets with time as a coordinate.

The derivative of a function $f$ with respect to $\beta$ can be found from its
Poisson bracket.

$$\dot{f} = \frac{df}{d\beta} = \sum_{\mu=0}^{D} \left( \frac{\partial f(q_{\nu}, p_{\nu})}{\partial q_{\mu}} \dot{q}_{\mu} + \frac{\partial f(q_{\nu}, p_{\nu})}{\partial p_{\mu}} \dot{p}_{\mu} \right)$$

$$= \sum_{i} \left( \frac{\partial f(q_{\nu}, p_{\nu})}{\partial q_{\mu}} \frac{\partial K(q_{\nu}, p_{\nu})}{\partial p_{\mu}} - \frac{\partial f(q_{\nu}, p_{\nu})}{\partial p_{\mu}} \frac{\partial K(q_{\nu}, p_{\nu})}{\partial q_{\mu}} \right) = \{f, K\}.$$

Therefore

$$\dot{f} = \{f, K\}.$$  \hspace{1cm} (5.11)

We can thus say that the quantity $f$ is a conserved quantity or constant of
motion with $\dot{f} = 0$, if and only if it has a vanishing Poisson bracket with
the extended Hamiltonian function $K$.

If phase-space functions $f(q_{\mu}, p_{\mu})$ and $g(q_{\mu}, p_{\mu})$ are constants of the motion,
then the phase-space function $\{f, g\}$ is also a constant of the motion.

Substituting $f$ in (5.12), we find

$$\dot{q}_{\mu} = \{q_{\mu}, K\} \quad \quad \quad \quad \dot{p}_{\mu} = \{p_{\mu}, K\}.$$  \hspace{1cm} (5.13)

Substituting $f$ and $g$ in (5.10), we get the following results

$$\{q_{\mu}, q_{\nu}\} = 0 \quad \quad \{q_{\mu}, p_{\nu}\} = \delta_{\mu\nu} \quad \quad \{p_{\mu}, p_{\nu}\} = 0$$  \hspace{1cm} (5.14)
For the zeroth components the equation is

\[ \{q_0, p_0\} = \{t, -H\} = -\{t, H\} = 1. \tag{5.15} \]

5.4 Promoting Poisson Brackets to Quantum Commutators

The commutators of the quantum operators have an algebraic structure that is similar to Poisson brackets. We can substitute Poisson brackets by commutators using the substitution \( \{,\} \rightarrow (\hbar i)^{-1}[,.] \). This substitution is called *canonical quantization*. [2] Using the Poisson Bracket results in (5.14) the commutators can be written as

\[
\begin{align*}
[q_{\mu}, q_{\nu}] &= 0 \\
[q_{\mu}, p_{\nu}] &= i\hbar \delta_{\mu \nu} \\
[p_{\mu}, p_{\nu}] &= 0
\end{align*}
\tag{5.16}
\]

The position operators in the Schrödinger theory are identical to classical variables in classical mechanics. Hence we can say that \( q_i \) is equivalent to \( q_i \).
6 Formalism in Quantum Mechanics

In this section we will develop a new approach of treating time in quantum mechanics based on the extended classical theory outlined above.

6.1 Deriving a Time-Translational Operator from the Commutation relationship

Let $\hat{A}$ and $\hat{B}$ be any two observables such that $[\hat{A}, \hat{B}] = i\hbar$. Using the Taylor series expansion, $f(x) = f(a) + f'(a)(x-a) + f''(a)/2!(x-a)^2 + f'''(a)/3!(x-a)^3 + ... + f^{(n)}(a)/n!(x-a)^n + ...$ on $f(\eta) = e^{\eta \hat{A}} \hat{B} e^{-\eta \hat{A}}$ about $\eta = 0$ where $d f / d\eta = [\hat{A}, \hat{f}]$, we can get

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + 1/2![\hat{A}, [\hat{A}, \hat{B}]] + 1/3![\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + ...$$

However, $[\hat{A}, \hat{B}] = i\hbar$. Then $[\hat{A}, [\hat{A}, \hat{B}]] = 0$. Therefore all other terms are equal to zero. Then $e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + i\hbar$. Multiplying $e^{\hat{A}}$ on both sides of this equation, we get $\hat{B} e^{-\hat{A}} = e^{-\hat{A}} \hat{B} + i\hbar e^{-\hat{A}}$. Now we can apply this on any ket, $|b\rangle$ where $\hat{B}|b\rangle = b|b\rangle$.

$$\hat{B} e^{-\hat{A}} |b\rangle = (b + i\hbar) e^{-\hat{A}} |b\rangle$$

Therefore, $(b + i\hbar)$ are the eigenvalues for the eigenstates $e^{-\hat{A}} |b\rangle$ under $\hat{B}$.

Now if $[\hat{A}', \hat{B}] = i\hbar$ and we let $\hat{A}' = i\alpha' / \hbar \hat{A}$, then $[\hat{A}, \hat{B}] = \alpha'$ as $i\alpha' / \hbar$ is a constant and we can take it out of the commutator. We can then write,
\[ \hat{B} e^{-i\theta / \hbar} |b\rangle = (b + i\hbar) e^{-i\theta / \hbar} |b\rangle. \] Then we can write,

\[ |b + i\hbar\rangle = e^{-i\theta / \hbar} |b\rangle. \]

Because of the unitary nature of the operator, \( e^{-i\theta / \hbar} |b\rangle \) has the same norm as \( |b\rangle \) therefore \( \langle b + i\hbar | b' + i\hbar \rangle = \langle b | b' \rangle \). Then \( e^{-i\theta / \hbar} \) is the generator of translation.

For any \( q'_0 \) we get a generator of time translation, \( e^{-i\theta q'_0 / \hbar} \).

### 6.2 Spacetime Representation

We can now find \( p_0, p_1 \) in the \( q_0, q_1 \) representation. We start by computing the \( q_0, q_1 - p_0, p_1 \) transformation function. We define \( |0_q\rangle \) the simultaneous eigenkets of \( q_0 \) and \( q_1 \) with eigenvalues \( q_\mu = 0 \), and \( |0_p\rangle \) the eigenkets of \( p_\mu \) with eigenvalues \( p_\mu = 0 \).

Then using the generator of translations we derived in the last section, we can write

\[ \langle q'_0, q'_1 | p'_0, p'_1 \rangle = \langle 0_q | e^{-i\theta q'_0 / \hbar} e^{i\theta q'_1 / \hbar} | p'_0, p'_1 \rangle. \] (6.1)

Then we can apply the translation operator on ket \( |p'_0, p'_1\rangle \) to get

\[ e^{-i\theta q'_0 / \hbar} e^{i\theta q'_1 / \hbar} \langle 0_q | p'_0, p'_1 \rangle. \]
Now the ket $|p'_1, p'_1\rangle$ can be written as $e^{-i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar} |0\rangle$. Therefore we get

$$
e^{-i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar} \langle q_0 | p'_0, p'_1 \rangle = e^{-i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar} \langle q_0 | e^{-i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar} |0\rangle$$

We can apply $e^{i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar}$ on $|0\rangle$ to get $e^{0} = 1$. Therefore we get

$$\langle q'_0, q'_1 | p'_0, p'_1 \rangle = e^{-i\varphi_0 \varphi_0/\hbar} e^{i\varphi_1 \varphi_1/\hbar} \langle q_0 |0\rangle \quad (6.2)$$

Now $\langle q_0 |0\rangle$ is a constant that can be determined using $\langle p'_0, p'_1 | p'_0, p'_1 \rangle = \delta(p'_0 - p'_0) \delta(p'_1 - p'_1)$. Using the completeness of the state $|q'_0, q'_1\rangle$, we know that

$$\int \int dq'_0 dq'_1 |q'_0, q'_1\rangle \langle q'_0, q'_1|0\rangle = 1.$$ So $\langle p'_0, p'_1 | p'_0, p'_1 \rangle = \int \int dq'_0 dq'_1 |p'_0, p'_1\rangle \langle p'_0, p'_1 | q'_0, q'_1\rangle \langle q'_0, q'_1 | p'_0, p'_1 \rangle$.

So we can write

$$\int \int dq'_0 dq'_1 |p'_0, p'_1\rangle \langle q'_0, q'_1|p'_0, p'_1\rangle = \delta(p'_0 - p'_0) \delta(p'_1 - p'_1) \quad (6.3)$$

Now to give a functional form for $\langle q'_0, q'_1 | p'_0, p'_1 \rangle$, we write the delta function as

$$\delta(p'_0 - p'_0) = \frac{1}{2\pi\hbar} \int dq'_0 e^{i(p'_0 - p'_0)q'_0/\hbar}$$

by integrating $e^{i(p'_0 - p'_0)q'_0/\hbar}$ over $q'_0$. Also,

$$\delta(p'_1 - p'_1) = \frac{1}{2\pi\hbar} \int dq'_1 e^{i(p'_1 - p'_1)q'_1/\hbar}$$

by integrating $e^{i(p'_1 - p'_1)q'_1/\hbar}$ over $q'_1$. 

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Combining the above two we get

\[ \delta(p'_0 - p''_0)\delta(p'_1 - p''_1) \]

\[ = \frac{1}{(2\pi\hbar)^2} \int \int dq'_0 dq'_1 e^{i(p'_0 - p''_0)q'_0/\hbar} e^{i(p'_1 - p''_1)q'_1/\hbar} \]

\[ = \frac{1}{(2\pi\hbar)^2} \int \int dq'_0 dq'_1 e^{i(p'_0 q'_0 + p'_1 q'_1)/\hbar} e^{-i(p''_0 q'_0 + p''_1 q'_1)/\hbar} \]

We can now compare this with eq. (6.3), to get

\[ \langle q'_0, q'_1 | p'_0, p'_1 \rangle = \frac{1}{2\pi\hbar} e^{i(p'_0 q'_0 + p'_1 q'_1)/\hbar} \]  

(6.4)

Now we consider an arbitrary ket, \(|a'\rangle\), of a system with one particle. The probability of finding the system at the point \(q'_0, q'_1\) in spacetime is given by \(\langle q'_0, q'_1 | a' \rangle\). Also the probability of finding the system at momenta \(p'_0, p'_1\) is \(\langle p'_0, p'_1 | a' \rangle\). The state \(|a'\rangle\) can be written in the coordinate basis as

\[ \langle q'_0 q'_1 | a' \rangle = \int \int dp'_0 dp'_1 \langle q'_0 q'_1 | p'_0 p'_1 \rangle \langle p'_0 p'_1 | a' \rangle \]  

(6.5)

\[ = \int \int dp'_0 dp'_1 e^{i(p'_0 q'_0 + p'_1 q'_1)/\hbar} \frac{1}{2\pi\hbar} \langle p'_0 p'_1 | a' \rangle \]  

(6.6)

where we use eq. (6.4) in the last step. We can translate this into wave mechanics as

\[ \psi_{a'}(q'_0, q'_1) = \frac{1}{2\pi\hbar} \int \int dp'_0 dp'_1 e^{i(p'_0 q'_0 + p'_1 q'_1)/\hbar} \phi_{a'}(p'_0, p'_1) \]  

(6.7)
where $\psi'_{\alpha'}(q_0',q_1') = \langle q_0'q_1'|a' \rangle$ is the wave function and $\phi'_{\alpha'}(p_0',p_1') = \langle p_0'p_1'|a' \rangle$ is the state $|a'\rangle$ in the momentum basis.

Now consider the action of the momentum operator on $\psi$. We get

$$\langle q_0'q_1'|\hat{p}_0 + \hat{p}_1|a' \rangle = \int \int dq_0'dq_1' \langle q_0'q_1'|\hat{p}_0 + \hat{p}_1|q_0''q_1'' \rangle \langle q_0''q_1''|a' \rangle. \quad (6.8)$$

We now consider $\langle q_0'q_1'|p_0 + p_1|q_0''q_1'' \rangle$. We can use the completeness of the state $|p_0'p_1'\rangle$ and write $\langle q_0'q_1'|p_0 + p_1|q_0''q_1'' \rangle$ as

$$\int \int dp_0'dp_1' \langle q_0'q_1'|\hat{p}_0 + \hat{p}_1|p_0',p_1' \rangle \langle p_0',p_1'|q_0''q_1'' \rangle$$

Now applying $\hat{p}_0 + \hat{p}_1$ on the state $|p_0',p_1'\rangle$ gives us

$$= \int \int dp_0'dp_1' \langle q_0'q_1'|p_0',p_1' \rangle \langle p_0' + p_1'|p_0',p_1'|q_0''q_1'' \rangle.$$

We can now use (6.4) to get

$$\frac{1}{(2\pi\hbar)^2} \int \int dp_0'dp_1'(p_0' + p_1')e^{i(p_0\delta_0 + p_1\delta_1 - p_0\delta_0' - p_1\delta_1')/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^2} \int \int dp_0'dp_1'(p_0' + p_1')e^{i(p_0\delta_0 + p_1\delta_1 - p_0\delta_0' - p_1\delta_1')/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^2} \int \int dp_0'dp_1'(p_0' + p_1')e^{i(p_0(\delta_0' - \delta_0) + p_1(\delta_1' - \delta_1))/\hbar}.$$
Therefore we can write

\[
\langle q_0'q_1'|p_0 + p_1|q_0''q_1'' \rangle = \frac{1}{(2\pi\hbar)^2} \int \int dp_0dp_1'(p_0' + p_1')e^{ip'(q' - q'')/\hbar}
\]

where we have used the notation \( p = (\hat{p}_0, \hat{p}_1) \). Now by the definition of the \( \delta \)-function [3], \( \delta(q' - q'') = \int \frac{dk}{2\pi} e^{ik(q' - q'')} \). Therefore \( \delta^{(1)}(q' - q'') = i \int \frac{dk}{2\pi} k e^{ik(q' - q'')} \). Now we let \( \overrightarrow{p}_0 = \hbar k_0 \) so that \( dp_0 = \hbar dk_0 \). Then we can write

\[
\langle q_0'q_1'|\overrightarrow{p}_0 + \overrightarrow{p}_1|q_0''q_1'' \rangle = \frac{1}{(2\pi\hbar)^2} \int \int \hbar \cdot dk_0dk_1'(k_0 + k_1')e^{i\hbar k_0(q_0'' - q_0') + ik_1'(q_1'' - q_1')}
\]

We can separate the integrals to get

\[
\frac{\hbar}{(2\pi)^2} \left[ \int \hbar \cdot dk_0'k_0'e^{i\hbar k_0'(q_0'' - q_0')} \int dk_1'e^{ik_1'(q_1'' - q_1')} \right]
\]

\[
+ \int dk_0'e^{i\hbar k_0'(q_0'' - q_0')} + \int dk_1'e^{ik_1'(q_1'' - q_1')}
\]

Now using the definition of the \( \delta \)-function, we can write

\[
\langle q_0'q_1'|\overrightarrow{p}_0 + \overrightarrow{p}_1|q_0''q_1'' \rangle = \frac{\hbar}{i} \left( \frac{\partial}{\partial q_0} \delta(q_0 - q_0'') \delta(q_1 - q_1') + \delta(q_0 - q_0'') \frac{\partial}{\partial q_0} \delta(q_1 - q_1') \right)
\]

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Therefore eq. (6.8) gives us

\[ \frac{\hbar}{i} \int \int dq_0'' dq_1'' \left( \frac{\partial}{\partial q_0} \delta(q_0' - q_0'') \delta(q_1' - q_1'') + \delta(q_0' - q_0'') \frac{\partial}{\partial q_1} \delta(q_1' - q_1'') \right) \langle q_0'' q_1'' | a' \rangle \]

We can break the integrals into two parts. This gives us

\[ \frac{\hbar}{i} \frac{\partial}{\partial q_0'} \int \int dq_0'' dq_1'' \delta(q_0' - q_0'') \delta(q_1' - q_1'') \langle q_0'' q_1'' | a' \rangle \]

\[ + \frac{\partial}{\partial q_1'} \int \int dq_0'' dq_1'' \delta(q_0' - q_0'') \delta(q_1' - q_1'') \langle q_0'' q_1'' | a' \rangle \]

Therefore we get

\[ \langle q_0' q_1' | \hat{p}_0 + \hat{p}_1 | a' \rangle = \frac{\hbar}{i} \left( \frac{\partial}{\partial q_0} \langle q_0' q_1' | a' \rangle + \frac{\partial}{\partial q_1} \langle q_0' q_1' | a' \rangle \right) \] (6.9)

This representation is very different from the Schrödinger picture. In the Schrödinger picture, a state \( |x, t\rangle \) means that we look at the position \( x \) of the system at a time \( t \). In this new approach, \( |q_0, q_1\rangle \) are simultaneous eigenstates of space and time. In this approach \( |\psi\rangle \) does not evolve with time as the time dependence does not lie in \( |\psi\rangle \). It must also be mentioned that this approach is different from the Heisenberg picture where the time dependence is in the operator.
7 History of Time in Quantum Mechanics

We will see how the problem of time arose in quantum mechanics by looking at the history of quantum mechanics. We will look at the work of Dirac, Heisenberg, Schrödinger, and Pauli.

7.1 Dirac

Dirac first began to deal with this problem in his article "Relativity quantum mechanics with an application to Compton scattering" [9]. He considers a dynamical system with \( n \)-variables where the Hamiltonian explicitly depends on time. He adds two extra variables, \( t \) and \(-W\) where \(-W\) is the conjugate momentum of \( t \). Here \( t \) is considered as an extra coordinate of the system, with minus the energy, \( W \) as its conjugate. So \( H(q_i, p_i, t) - W = 0 \). (Later in this paper, we will denote \(-W\) as \( p_0 \).) Then in quantum mechanics, \( t \) and \(-W\) commute with each other, that is to say, \( tW - Wt = -i\hbar \). But now since \( H \) does not depend on \( W \), \( H \) commutes with time. Dirac sees this problem but avoids it by using a simple system where he follows the classical approach very closely.

In his article, "On the theory of quantum mechanics"[10], he abandons this approach and reverts back to Schrödinger's wave mechanics. He starts with the Schrödinger equation, \( \left\{ H(q_i, i\hbar \partial / \partial q_i) - W \right\} \psi = 0 \). This is closely connected to the earlier Hamiltonian equation, \( H(q_i, p_i, t) - W = 0 \). He treats \(-W\) as a differential operator like \( p_i \). \(-W = -i\hbar \partial / \partial t\).
Dirac struggles with his notation as he later switches from $q$ to $x$ for the spacial coordinates in his papers. This confusion is caused by the use of $q_i$ to represent space in classical mechanics and $x_i$ to represent space in relativity. In his book, “Quantum Mechanics” [11], Dirac derives that in the $x$-representation, that is a state $\psi$ is given by the wave function $(x|\psi)$, $d_x = -\partial/\partial x$ and $d_xx - xd_x = -1$. Using this and the canonical commutation, he derives, $p_x = -i\hbar d_x$. The same can be done for $y$ and $z$. Dirac goes on to consider the operation of the time-displacement operator on the state $\psi$. He shows that $d_t = -\partial/\partial t$. However to get the correct equations of motion we must put $H = -i\hbar d_t$, where $H$ is the Hamiltonian. therefore everything works except there is a change of sign compared to $p_x = i\hbar d_x$. Dirac notices the difference but gives no explanation.

7.2 Heisenberg

Heisenberg discussed the role of time in the “three-men-paper” (Born, Heisenberg and Jordan 1926)[12], often regarded as the beginning of modern quantum mechanics. There is no commutation relation for time that is given. Time figures in as an ordinary parameter in a time-dependent Hamiltonian. In a review article written a month after the paper was published, Heisenberg points out that time is essentially treated differently from the spacial coordinates thus the theory should still be considered incomplete.
7.3 Schrödinger

Schrödinger begins by setting up spacetime coordinates in quantum mechanics as required by relativity. He uses light signals to synchronize a clock. He realized that to get an accuracy of \( \tau \) in time, the frequency spread of the light must at least be of the order \( \frac{1}{4\pi\tau} \). Therefore there is an uncertainty in the momentum transferred to the clock and thus an uncertainty in the velocity. Schrödinger realized that the uncertainty in setting up the clock of mass \( m \) is no less than \( \frac{\hbar}{4\pi\tau} \). Therefore, for the spacetime coordinate system to be accurate, we must use very massive clocks. Therefore Schrödinger argues that from the perspective of quantum mechanics, special relativity should be considered as a macroscopic approximation like classical mechanics. Therefore formulas, such as the Lorentz transformation, from relativity need to be quantized before they can be used in quantum mechanics.

However quantum mechanics does not treat time and space equally. Time in quantum mechanics is a \( c \)-number. Schrödinger considers this a very problematic issue in quantum mechanics. Knowledge about time is acquired like any other variable that is by observing a physical system, in this case a clock. Therefore time should be treated as an observable. Schrödinger considers two clocks: one that defines the time appearing in the wave function and the other that is described by the former. For this clock to be ideal, wave functions given by different values of \( t \) need to be orthogonal. Schrödinger proves that this is only true if the energy of the clock is completely uncertain; that is to say that all values of energy are equally probable. He therefore concludes
that such a state of a system is meaningless.

7.4 Pauli

Pauli considers Schrödinger's argument that a system where time is completely known will have infinite uncertainty for energy. Pauli however believed that such a system can be approximated. Pauli gave an argument against the existence of a self-adjoint time operator in a footnote in his Encyclopedia of Physics article. He argued that if there existed a self-adjoint time operator $\hat{T}$, such that $[\hat{H}, \hat{T}] = -i\hbar$, then the application of the unitary operator, $e^{-iE_1\hat{T}/\hbar}$ to the eigenstate $|E\rangle$ would produce new energy eigenstates with eigenvalues $E - E_1$, so that the spectrum of $E$ would necessarily have to extend continuously over the range $[-\infty, \infty]$. Most physical systems of interest have bounded, semi-bounded or discrete Hamiltonians which means that they must have some lowest state. Therefore this condition leaves no room for a self-adjoint time operator.
8 Applications

We will now discuss some of the applications of this new approach which is different from the ones described above. Using this new approach of handling time in quantum mechanics we can now derive fundamental results.

8.1 Uncertainty Principle

We begin with the Schwarz inequality, \( \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2 \). [2] Let \( \hat{A} \) and \( \hat{B} \) be two observables. Now in the above equation we let \( \Delta \hat{A}|\psi\rangle = |\alpha\rangle \) and \( \Delta \hat{B}|\psi\rangle = |\beta\rangle \). \( \Delta \hat{A} \) and \( \Delta \hat{B} \) can be applied to any ket \( |\psi\rangle \). We can define \( \Delta \hat{A} \) by \( (\Delta \hat{A})^2 = \hat{A}^2 - \langle \hat{A} \rangle^2 \). Therefore we get

\[
\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2
\]

because \( \Delta \hat{A} \) and \( \Delta \hat{B} \) are Hermitian. \( \Delta \hat{A} \Delta \hat{B} \) can be written as

\[
1/2[\Delta \hat{A}, \Delta \hat{B}] + 1/2\{\Delta \hat{A}, \Delta \hat{B}\}
\]

where \( \{\Delta \hat{A}, \Delta \hat{B}\} \) is the anticommutator. So

\[
1/2[\Delta \hat{A}, \Delta \hat{B}] + 1/2\{\Delta \hat{A}, \Delta \hat{B}\}
\]

\[
= 1/2(\Delta \hat{A} \Delta \hat{B} - \Delta \hat{B} \Delta \hat{A}) + 1/2(\Delta \hat{A} \Delta \hat{B} + \Delta \hat{B} \Delta \hat{A})
\]

\[
= \Delta \hat{A} \Delta \hat{B}.
\]
Now $[\Delta \hat{A}, \Delta \hat{B}]$ is anti-Hermitian as

$$([\Delta \hat{A}, \Delta \hat{B}])^\dagger = (\Delta \hat{A} \Delta \hat{B} - \Delta \hat{B} \Delta \hat{A})^\dagger$$

$$= \Delta \hat{B} \Delta \hat{A} - \Delta \hat{A} \Delta \hat{B} = -[\Delta \hat{A}, \Delta \hat{B}]$$

Also the anticommutator $\{\Delta \hat{A}, \Delta \hat{B}\}$ is Hermitian. Therefore $\langle \Delta \hat{A} \Delta \hat{B} \rangle = 1/2\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle + 1/2\{\Delta \hat{A}, \Delta \hat{B}\}$. On the right hand side, the first part is purely imaginary because the expectation value of an anti-Hermitian operator is purely imaginary. Also, the right part is purely real. This is because the expectation value of a Hermitian operator is purely real. Therefore

$$Re[\langle (\Delta \hat{A} \Delta \hat{B}) \rangle^2] = 1/4(\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle^2 + 1/4(\langle \{\Delta \hat{A}, \Delta \hat{B}\} \rangle))$$

Then

$$|\langle (\Delta \hat{A} \Delta \hat{B}) \rangle|^2 = 1/4|\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle|^2 + 1/4|\langle \{\Delta \hat{A}, \Delta \hat{B}\} \rangle|^2$$

We know that $\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geq |\langle \Delta \hat{A} \Delta \hat{B} \rangle|^2$, therefore

$$\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geq 1/4|\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle|^2. \quad (8.1)$$

This inequality is the uncertainty relation.

This uncertainty relation can be applied to $q_0$ and $p_0$. We know that the
commutator relationship \([q_0, p_0] = i\hbar\). Then we can write

\[
\langle (\Delta q_0)^2 \rangle \langle (\Delta p_0)^2 \rangle \geq \frac{1}{4} |i\hbar|^2
\]

(8.2)

\[
\geq \frac{1}{4} |i\hbar|^2
\]

(8.3)

\[
\geq \frac{\hbar^2}{4}
\]

(8.4)

This gives us the \(q_0 - p_0\) uncertainty relation.

### 8.2 Schrödinger's equation

We can obtain the Schrödinger's equation in the new formalism by looking at the extended Hamiltonian, \(\hat{K}\) which is given by \(\hat{p}_0 + \hat{H}\) where \(\hat{H}\) is the standard Hamiltonian. Now consider

\[
\langle q_0, q_1 | \hat{K} | \psi \rangle = \langle q_0, q_1 | \hat{p}_0 + \hat{H} | \psi \rangle
\]

Then by the action of \(p_0\) on \(\langle q_0, q_1 \rangle\), we can write

\[
\langle q_0, q_1 | \hat{p}_0 + \hat{H} | \psi \rangle = i\hbar \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle + H \langle q_0, q_1 | \psi \rangle
\]

However by definition, \(\hat{K} | \psi \rangle = 0\), therefore

\[
i\hbar \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle + H \langle q_0, q_1 | \psi \rangle = 0
\]
Then \( \frac{\hbar}{i} \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle = H \langle q_0, q_1 | \psi \rangle \). This gives us the Schrödinger's equation.

For the case of the simple harmonic oscillator, we get \( \hat{H} = \frac{\hat{p}_1^2}{2m} + V(\hat{x}) \).

Therefore we get

\[
\frac{\hbar}{i} \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle = \frac{\langle \partial \hat{q}_1 \rangle^2}{2m} \langle q_0, q_1 | \psi \rangle + V(\hat{x}) \langle q_0, q_1 | \psi \rangle.
\]

We can write this in wave mechanics as

\[
\frac{\hbar}{i} \frac{\partial}{\partial q_0} \psi(q_0, q_1) = \frac{1}{2m} \left( \frac{\partial}{\partial q_1} \right)^2 \psi(q_0, q_1) + V(\hat{x}) \psi(q_0, q_1).
\]

### 8.3 Ehrenfest Theorem

We start in Schrödinger's picture and translate it to the new approach. For any arbitrary observable \( \hat{A} \), in the Schrödinger picture,

\[
\frac{\partial}{\partial t} \langle \hat{A} \rangle = \frac{\partial}{\partial t} \int \psi^*(x, t) \hat{A}(x, t) \psi(x, t) dx
\]

In the new formalism, we write this as

\[
= \frac{\partial}{\partial q_0} \int \langle \psi | q_0, q_1 \rangle \langle q_0, q_1 | \hat{A}(q_0, q_1) | \psi \rangle dq_1.
\]

We can take the partial derivative inside the integral as we are integrating over \( q_1 \). Using the chain rule, we get

\[
\frac{\partial}{\partial t} \langle \hat{A} \rangle = \int \frac{\partial}{\partial q_0} \langle \psi | q_0, q_1 \rangle \langle q_0, q_1 | \hat{A}(q_0, q_1) | \psi \rangle dq_1.
\]
+ \int \langle \psi | q_0, q_1 \rangle \frac{\partial}{\partial q_0} \langle q_0, q_1 | \hat{A}(q_0, q_1) | \psi \rangle dq_1

Now we can write $\frac{\partial}{\partial q_0} \langle q_0, q_1 | \hat{A}(q_0, q_1) | \psi \rangle$ as $\frac{\partial}{\partial q_0} \{ A(q_0, q_1) \langle q_0, q_1 | \psi \rangle \}$ where $A(q_0, q_1)$ is a c-number. Using the chain rule, $\frac{\partial}{\partial q_0} \{ A(q_0, q_1) \langle q_0, q_1 | \psi \rangle \} = \frac{\partial A(q_0, q_1)}{\partial q_0} \langle q_0, q_1 | \psi \rangle + A(q_0, q_1) \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle$. Then we have

$$\frac{\partial}{\partial t} \langle \hat{A} \rangle = \int \left[ \frac{\partial}{\partial q_0} \langle \psi | q_0, q_1 \rangle \right] A(q_0, q_1) \langle q_0, q_1 | \psi \rangle dq_1$$

$$+ \int \langle \psi | q_0, q_1 \rangle \frac{\partial A(q_0, q_1)}{\partial q_0} \langle q_0, q_1 | \psi \rangle dq_1 + \int \langle \psi | q_0, q_1 \rangle A(q_0, q_1) \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle dq_1.$$

We know that the action of $\hat{a}$ on $\langle q_0, q_1 | \psi \rangle$ gives us $\frac{\hbar}{i} \frac{\partial}{\partial q_0} \langle q_0, q_1 | \psi \rangle$, therefore we can write

$$\frac{\partial}{\partial t} \langle \hat{A} \rangle = \int \left[ \frac{-i}{\hbar} \langle \psi | q_0, q_1 \rangle \right] A(q_0, q_1) \langle q_0, q_1 | \psi \rangle dq_1$$

$$+ \int \langle \psi | q_0, q_1 \rangle \frac{\partial A(q_0, q_1)}{\partial q_0} \langle q_0, q_1 | \psi \rangle dq_1 + \int \frac{i}{\hbar} \langle \psi | q_0, q_1 \rangle A(q_0, q_1) \langle q_0, q_1 | \hat{a} \rangle dq_1 \langle q_0, q_1 | \psi \rangle dq_1$$

This gives us the Ehrenfest theorem in the new approach.

### 8.4 Propagator

In the new approach, the propagator is $\langle q_0, q_1 | \beta | q_0', q_1' \rangle$, where $\beta$ is the monotonous increasing parameter. For $\beta = \beta'$, we get $\langle q_0, q_1, \beta | q_0', q_1' \rangle = \delta(q_0 - q_0') \delta(q_1 - q_1')$. This is because $q_0, q_1$ and $q_0', q_1'$ vary with $\beta$ and $\beta'$. When $\beta = \beta'$, the inner product of $| q_0, q_1 \rangle$ with $| q_0', q_1' \rangle$ is zero unless $q_0 = q_0'$.
and \( q_1 = q'_1 \).

To consider the case of \( \beta \neq \beta' \), we first have to define the beta-evolution operator \( U \). We want to know how any arbitrary state ket changes as \( \beta \) changes. Therefore two kets are related by \( |\alpha, \beta\rangle = U(\beta, \beta_0)|\alpha, \beta_0\rangle \). We define the operator to be unitary, that is \( U^\dagger(\beta, \beta_0)U(\beta, \beta_0) = 1 \). This ensures that if the initial state ket is normalized, all state kets at a later beta are also normalized. We also require that \( U(\beta_2, \beta_0) = U(\beta_2, \beta_1)U(\beta_1, \beta_0) \).

Now let us consider the infinitesimal beta-evolution operator, \( U(\beta_0 + d\beta, \beta_0) \). So \( |\alpha, \beta_0 + d\beta\rangle = U(\beta_0 + d\beta, \beta_0)|\alpha, \beta_0\rangle \). Then \( \lim_{d\beta \to 0} U(\beta_0 + d\beta, \beta_0) = 1 \). Then the difference between 1 and the infinitesimal operator is of the order of \( d\beta \).

Now from the classical extended theory, we know that the generator of beta evolution in the extended Hamiltonian \( K \). Therefore \( U(\beta_0 + d\beta, \beta_0) = 1 - \frac{iK d\beta}{\hbar} \).

Let us consider now \( U(\beta + d\beta, \beta_0) = U(\beta + d\beta, \beta)U(\beta, \beta_0) \). We know this from our assumption about \( U \). Then

\[
U(\beta + d\beta, \beta)U(\beta, \beta_0) = (1 - \frac{iK d\beta}{\hbar})U(\beta, \beta_0).
\]

Expanding the expression on the right hand side, we get

\[
U(\beta + d\beta, \beta_0) - U(\beta, \beta_0) = -\frac{iK d\beta}{\hbar}U(\beta, \beta_0).
\]

By definition, we can write this as a differential equation

\[
\frac{i\hbar}{\partial \beta} U(\beta, \beta_0) = KU(\beta, \beta_0)
\]

(8.5)
For the case that $\mathcal{K}$ is independent of $\beta$, the solution to the above equation is given by

$$U(\beta, \beta_0) = e^{-\frac{i\mathcal{K}}{\hbar}(\beta-\beta_0)}$$

If we let $\beta_0 = 0$, we can write $U(\beta, \beta_0 = 0) = e^{-i\mathcal{K} \beta}$. Now let's come back to the question of the propagator, $\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle$, when $\beta \neq \beta'$. By summing over the eigenstates of $\mathcal{K}$, we can write $\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle$ as $\sum_{n_k} \langle q_0, q_1, \beta | n_k \rangle \langle n_k | q'_0, q'_1, \beta' \rangle$. Using the beta evolution operator above, we can now write

$$\sum_{n_k} \langle q_0, q_1, \beta | n_k \rangle \langle n_k | q'_0, q'_1, \beta' \rangle = \sum_{n_k} \langle q_0, q_1, 0 | e^{-\frac{i\mathcal{K}}{\hbar} n_k} | n_k \rangle \langle n_k | e^{-\frac{i\mathcal{K}}{\hbar} q'_0, q'_1, 0} \rangle.$$

Assuming that $\mathcal{K} | n_k \rangle = K_{n_k} | n_k \rangle$ for any arbitrary ket $| n_k \rangle$, this becomes

$$\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \sum_{n_k} e^{-iK_{n_k}(\beta-\beta')} \langle q_0, q_1, 0 | n_k \rangle \langle n_k | q'_0, q'_1, 0 \rangle.$$

This result is the propagator in the new formalism. This can be further simplified by noting that $\mathcal{K} | n_k \rangle = (\hat{p}_0 + \hat{H}) | n_k \rangle = 0$, therefore all the eigenvalues $K_{n_k}$ are zero. We then can

$$\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \sum_{n_k} \langle q_0, q_1, 0 | n_k \rangle \langle n_k | q'_0, q'_1, 0 \rangle.$$

We assume that the eigenstates of $\mathcal{K}$ are normalized so that $\sum_k | n_k \rangle \langle n_k | = 1$. Then $\langle q_0, q_1 | n_k \rangle = \psi_n(q_1) e^{iE_n q_0}$. 40
Therefore

$$\langle q_0, q_1, \beta | q'_0, q'_1, \beta' \rangle = \sum_n \psi_n(q_1) e^{\frac{i}{\hbar} E_n q_0} \psi^*_n(x) e^{-\frac{i}{\hbar} E_n q_0}$$

which is the standard form of the propagator.

## 8.5 Probability of a state $|\psi\rangle$ at $q_0$

Let $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ where $a_0 = a_1 = \frac{1}{\sqrt{2}}$. We can write this in the $q_0$ basis as

$$\langle q_0 | \psi \rangle = a_0 \langle q_0 | 0 \rangle + a_1 \langle q_0 | 1 \rangle$$

We know that $\langle q_0 | 0 \rangle = e^{-i \frac{E_0 q_0}{\hbar}}$ and $\langle q_0 | 1 \rangle = e^{-i \frac{E_1 q_0}{\hbar}}$. Then

$$\langle q_0 | \psi \rangle = a_0 e^{-i \frac{E_0 q_0}{\hbar}} + a_1 e^{-i \frac{E_1 q_0}{\hbar}}$$

Now the probability of the state $|\psi\rangle$ at the coordinate $q_0$ is $|\langle q_0 | \psi \rangle|^2$. Therefore multiplying $\langle q_0 | \psi \rangle$ with its conjugate, we get $1 + \cos[\frac{2\pi}{\hbar}(E_1 - E_0)]$. The graph of this sinusoidal function for 1.5 periods is as follows.
Figure 1: The probability of finding a particle in the state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ at $q_0$ (where $a_0 = a_1 = \frac{1}{\sqrt{2}}$, $E_0 = 1$, $E_1 = 2$ and $\hbar$ has been set equal to 1.)

9 Conclusion

In conclusion, we were able to develop a formalism that treats time at an equal footing as the spacial coordinates. We were able to derive results in quantum theory using this formalism. This includes the Schrödinger equation, Ehrenfest theorem and the propagator. We also saw how the formalism is different from the way time has been treated in quantum mechanics through a brief history of time in quantum mechanics and its treatment by some of the founders of modern quantum theory.
References


